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Stringy instanton counting and topological strings

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ABSTRACT: We study the stringy instanton partition function of four dimensional $\mathcal{N} = 2$ $U(N)$ supersymmetric gauge theory which was obtained by Bonelli et al. in 2013. In type IIB string theory on $\mathbb{C}^2 \times T^*\mathbb{P}^1 \times \mathbb{C}$, the stringy $U(N)$ instantons of charge k are described by k D1-branes wrapping around the \mathbb{P}^1 bound to N D5-branes on $\mathbb{C}^2 \times \mathbb{P}^1$. The KK corrections induced by compactification of the \mathbb{P}^1 give the stringy corrections. We find a relation between the stringy instanton partition function whose quantum stringy corrections have been removed and the K-theoretic instanton partition function, or by geometric engineering, the refined topological A-model partition function on a local toric Calabi-Yau threefold. We also study the quantum stringy corrections in the stringy instanton partition function which is not captured by the refined topological strings.

KEYWORDS: Topological Strings, Supersymmetric gauge theory, Topological Field Theories

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1 Introduction

In 1994, by Seiberg and Witten the exact prepotential $\mathcal{F}_0 = \mathcal{F}_0^{\text{pert}} + \mathcal{F}_0^{\text{inst}}$ in four dimensional $\mathcal{N} = 2$ $SU(2)$ supersymmetric gauge theory was obtained [1, 2], where $\mathcal{F}_0^{\text{pert}}$ and $\mathcal{F}_0^{\text{inst}}$ are the perturbative and instanton part of the prepotential, respectively. The prepotential is computed from a period of a two dimensional algebraic curve on the Coulomb branch which is called Seiberg-Witten curve. This exact result can be generalized to other gauge theories with ADE gauge symmetries. By the compactification of type IIA strings on

a local Calabi-Yau threefold given by ALE space fibration of ADE type over \mathbb{P}^1 , one obtains $\mathcal{N} = 2$ supersymmetry with ADE gauge symmetry in four dimensions. This realization is known as geometric engineering of gauge theory [3–5]. Then the Seiberg-Witten curve is embedded into the mirror dual of the local Calabi-Yau threefold. In 2002, by Nekrasov the prepotential \mathcal{F}_0 was directly derived from path integral formulation using the localization technique [6]. A necessary ingredient of this computation is to introduce the Omega background described by $\mathbb{R}^4 \cong \mathbb{C}^2$ fibration over two dimensional torus T^2 . The Omega background has two generators ϵ_1 and ϵ_2 of T^2 . The instanton moduli space of the gauge theory can be described by the ADHM moduli space whose dynamical variables are given by matrices [7] (see (2.1)). The computation of the path integral which gives the instanton partition function $Z^{\text{Nek}}(\epsilon_1, \epsilon_2)$ on the Omega background reduces to the computation of the equivariant volume of the ADHM moduli space whose IR behavior was regularized by $\epsilon_{1,2}$ [6, 8, 9].¹ By the localization, the instanton partition function $Z^{\text{Nek}}(\epsilon_1, \epsilon_2)$ can be exactly computed, and one obtains the asymptotic expansion of the form

$$\log Z^{\text{Nek}}(\epsilon_1, \epsilon_2) = \sum_{g, \ell=0}^{\infty} (\epsilon_1 \epsilon_2)^{g-1} (\epsilon_1 + \epsilon_2)^{\ell} \mathcal{F}_{g, \ell}^{\text{Nek}}.$$

Here the leading term coincides with the instanton part of the prepotential $\mathcal{F}_{0,0}^{\text{Nek}} = \mathcal{F}_0^{\text{inst}}$ [6, 10–12]. The $SU(N)$ instanton partition function on the anti-self-dual Omega background $\epsilon_1 = -\epsilon_2 = \hbar$ can be also obtained from a geometric engineering limit of the topological A-model partition function on a local toric Calabi-Yau threefold given by ALE space fibration of A_{N-1} type over \mathbb{P}^1 [13–18]. Here the topological A-model partition function is computed by the topological vertex formalism [19, 20], and the parameter \hbar is identified with the topological string coupling constant g_s . In [21, 22], refinements of the topological vertex formalism were proposed so that the $SU(N)$ instanton partition function on the general Omega background is obtained [22–24].

In 2012, the partition function of $\mathcal{N} = (2, 2)$ gauged linear sigma model (GLSM) on S^2 was exactly computed [25, 26]. By using this result, the stringy instanton partition function of four dimensional $\mathcal{N} = 2$ $U(N)$ supersymmetric gauge theory was given in [27]. The GLSM, which gives the stringy instanton partition function, flows in the IR fixed point to an $\mathcal{N} = (2, 2)$ non-linear sigma model (NLSM) whose target space is given by the ADHM moduli space. In [27], it was shown that in the degenerate limit of the worldsheet S^2 the stringy instanton partition function yields the $U(N)$ instanton partition function $Z_N^{\text{Nek}}(\epsilon_1, \epsilon_2)$ (see (2.12) for the k -instanton sector).² Here $Z_N^{\text{Nek}}(\epsilon_1, \epsilon_2)$ does not depend on the Kähler modulus ζ of the ADHM moduli space, whereas the stringy instanton partition function depends on ζ . In section 2 we review the results in [27], and discuss the formal structures of the stringy $U(N)$ instanton partition function.

¹The instanton partition function $Z^{\text{Nek}}(\epsilon_1, \epsilon_2)$ also depends on the Coulomb moduli described by the Cartan subalgebra for gauge group, the dynamical scales in four dimensional gauge theory, and the masses of matters if the gauge theory is coupled to them. In this introduction we abbreviate these arguments for simplicity.

²The stringy instanton counting in this paper means the “gauge theoretic” instanton counting with the stringy corrections. This is different from “stringy” (or “exotic”) instantons discussed in e.g. [28–38].

The stringy instanton partition function has the classical stringy corrections and quantum stringy (α') corrections. In the anti-self-dual Omega background $\epsilon_1 = -\epsilon_2 = \hbar$, the stringy instanton partition function does not have the quantum stringy corrections, and only have the classical stringy corrections [27]. In section 3, we study the classical part $Z_N^{\text{cSI}}(\epsilon_1, \epsilon_2; \zeta)$ of the stringy $U(N)$ instanton partition function (see (2.17) for the k -instanton sector), and show that $Z_N^{\text{cSI}}(\epsilon_1, \epsilon_2; \zeta)$ is reduced from a four dimensional limit of the K-theoretic (“five dimensional $\mathbb{R}^4 \times S^1$ ”) $U(N)$ instanton partition function $Z_{N,m}^{\text{K-Nek}}(\epsilon_1, \epsilon_2)$ with five dimensional Chern-Simons coefficient $m \in \mathbb{Z}$ [39, 40]. By geometric engineering, the classical stringy instanton partition function $Z_N^{\text{cSI}}(\epsilon_1, \epsilon_2; \zeta)$ is also reduced from the refined topological A-model partition function $Z_{N,m}^{\text{refA}}(\epsilon_1, \epsilon_2)$ on a family of local toric Calabi-Yau threefolds with resolved A_{N-1} singularity labeled by $m \in \mathbb{Z}$ described in figure 2 (section 3.2). Then we obtain a relation

$$Z_{N,m}^{\text{refA}}(\epsilon_1, \epsilon_2) \sim Z_{N,m}^{\text{K-Nek}}(\epsilon_1, \epsilon_2) \xrightarrow{\text{4d limit}} Z_N^{\text{cSI}}(\epsilon_1, \epsilon_2; \zeta) \xrightarrow{r \rightarrow 0} Z_N^{\text{Nek}}(\epsilon_1, \epsilon_2).$$

Here the four dimensional (4d) limit is given by

$$\beta \rightarrow 0, \quad m \rightarrow \infty \quad \text{with fixed} \quad \beta m \sim \zeta r,$$

where β is the radius of the five dimensional circle S^1 , and r is the radius of the worldsheet S^2 . The limit $r \rightarrow 0$ corresponds to the degenerate limit of the S^2 . In section 3.3, we give a physical explanation of the above relation by revisiting string dualities discussed in [41]. Then this relation claims geometric engineering of the instantons with the classical stringy corrections, and shows that the Kähler modulus ζ of the ADHM moduli space is obtained from the strong coupling limit of the five dimensional Chern-Simons term, and quantization of ζ can be interpreted as a five dimensional Chern-Simons coefficient $m \in \mathbb{Z}$.

It was conjectured in [42] and proved in [43] that the $\mathcal{N} = (2, 2)$ GLSM partition function Z_X^{GLSM} on S^2 which flows in the IR fixed point to an $\mathcal{N} = (2, 2)$ NLSM on S^2 whose target space is a Calabi-Yau geometry X gives the quantum-corrected Kähler potential K_X on the Kähler moduli space of X :

$$e^{-K_X} = Z_X^{\text{GLSM}}.$$

Therefore the stringy $U(N)$ instanton partition function gives the Kähler potential for the ADHM moduli space. Note that the ADHM moduli space has a hyper-Kähler structure [44], and thus satisfies the Calabi-Yau condition. In [27], it was proposed that the quantum stringy corrections gives us the Givental’s \mathcal{I} -function [45] (see also [46]) of the ADHM moduli space. In section 4, for $U(1)$, $U(2)$, and $U(3)$ we study the full stringy instanton partition functions with the quantum stringy corrections. For each N , we find a universal structure of the quantum stringy corrections for arbitrary instanton charge k . For $U(1)$, as discussed in [27], we confirm agreements with quantum correlators in the T -equivariant cohomology ring $H_T^*(\text{Hilb}_k(\mathbb{C}^2), \mathbb{Q})$ (with the equivariant parameters $\epsilon_{1,2}$) of the Hilbert scheme $\text{Hilb}_k(\mathbb{C}^2)$ of points on \mathbb{C}^2 . For $U(2)$ and $U(3)$, using the formulation of Maulik and Okounkov [47], we also check agreements with the quantum parts of quantum correlators in the equivariant cohomology of the ADHM moduli space. In section 5 we give

our conclusions and discuss some future directions. Appendix A is a note on the multiple gamma function which appears in this paper. In appendix B, we discuss relations between the stringy U(1) instanton partition function, the simple Hurwitz theory, and the topological A-model on a local toric Calabi-Yau threefold. Here a relation between the latter two theories was previously discussed in [48]. In appendix C, we review the Fock space description of $H_T^*(\text{Hilb}_k(\mathbb{C}^2), \mathbb{Q})$ [44, 49–51], and compute some equivariant correlators for comparing with the stringy U(1) instanton partition function. In appendix D, we review and discuss the higher rank generalization of the formulation in appendix C [47, 52]. In appendix E, we summarize the formulas of the exact Kähler potentials on quantum Kähler moduli spaces of Calabi-Yau threefolds (e.g. [42]) and fourfolds (conjectured in [53]).

2 Stringy U(N) instanton partition function

In this section, we review the stringy U(N) instanton partition function given in [27], and discuss its formal structures.

2.1 $\mathcal{N} = (2, 2)$ GLSM on S^2 for ADHM moduli space

Let us consider type IIB strings on $\mathbb{C}^2 \times T^*\mathbb{P}^1 \times \mathbb{C}$, and introduce D1-D5 brane system consisting of N D5-branes on $\mathbb{C}^2 \times \mathbb{P}^1$ and k D1-branes wrapping around the \mathbb{P}^1 . By embedding the U(1) spin connection on \mathbb{P}^1 into the SO(4) R-symmetry in the world volume theory of the N D5-branes,³ and by compactifying this theory on \mathbb{C}^2 the four dimensional $\mathcal{N} = 2$ U(N) supersymmetric gauge theory is obtained [54] (see also e.g. [55]). Here the k D1-branes describe the instantons of charge k in the four dimensional gauge theory.⁴ As the world volume theory on the k D1-branes, these instantons are described by an $\mathcal{N} = (2, 2)$ NLSM on \mathbb{P}^1 whose target space is the ADHM (framed k -instanton) moduli space⁵

$$\mathcal{M}_{k,N} = \{(B_1, B_2, I, J) | [B_1, B_2] + IJ = 0, \\ [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \zeta \mathbf{1}_{k \times k}\} / \text{U}(k). \quad (2.1)$$

Here $B_{1,2}: \mathbb{C}^k \rightarrow \mathbb{C}^k$, $I: \mathbb{C}^N \rightarrow \mathbb{C}^k$, $J: \mathbb{C}^k \rightarrow \mathbb{C}^N$, and $\zeta > 0$ defines the Kähler modulus. The gauge transformation is given by $(B_1, B_2, I, J) \mapsto (R^{-1}B_1R, R^{-1}B_2R, R^{-1}I, JR)$, where $R \in \text{U}(k)$. This NLSM is obtained in the IR fixed point of an $\mathcal{N} = (2, 2)$ U(k) GLSM on $S^2 \cong \mathbb{P}^1$ with the matter content described in table 1 [27]. The twisted masses $\epsilon_{1,2} \in \mathbb{R}$ give the generators of $T^2 = \text{U}(1)^2$ which rotates the \mathbb{C}^2 in $\mathbb{C}^2 \times \mathbb{P}^1$, and these masses induce the Omega background. The twisted masses $a_\alpha \in \mathbb{R}$ give the generators of the Cartan subalgebra of U(N). These chiral fields interact each other through a superpotential $W = \text{Tr}_k \chi([B_1, B_2] + IJ)$ whose total U(1)_V R-charge is two. The Fayet-Iliopoulos (FI)

³This embedding breaks the R-symmetry as $\text{SO}(4) \cong \text{SU}(2) \times \text{SU}(2)$ to $\text{U}(1) \times \text{SU}(2)$.

⁴This theory has the KK corrections by the compactification of the \mathbb{P}^1 , and these corrections are given as stringy corrections. By the degeneration of \mathbb{P}^1 , the D1-D5 brane system becomes the fractional D(−1)-D3 brane system at the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold singularity [56–58] which describes the four dimensional $\mathcal{N} = 2$ U(N) supersymmetric gauge theory without the KK corrections.

⁵The ADHM moduli space has a hyper-Kähler structure and the complex dimension is given by $\dim_{\mathbb{C}} \mathcal{M}_{k,N} = 2kN$ [44].

Field	$U(k)$	Twisted mass	$U(1)_V$
χ	adj.	ϵ	$2 - 2\mathfrak{q}$
$B_{1,2}$	adj.	$-\epsilon_{1,2}$	\mathfrak{q}
I_α	k	$-a_\alpha$	$\mathfrak{q} + \mathfrak{p}$
J_α	$\bar{\mathbf{k}}$	$a_\alpha - \epsilon$	$\mathfrak{q} - \mathfrak{p}$

Table 1. Matter content of the GLSM for the ADHM moduli space $\mathcal{M}_{k,N}$. Here $\epsilon = \epsilon_1 + \epsilon_2$ and $\alpha = 1, \dots, N$. By restricting the $U(1)_V$ R-charges to be non-negative, these are constrained as $0 < \mathfrak{p} < \mathfrak{q} < 1$.

parameter for central $U(1) \subset U(k)$ gives the Kähler modulus ζ in the ADHM moduli space (2.1).

By using the formula of the GLSM partition function on S^2 obtained by the supersymmetric localization [25, 26], after taking the limit $\mathfrak{p}, \mathfrak{q} \rightarrow 0^+$ due to the non-compactness of the $\mathcal{M}_{k,N}$ [59], one obtains the stringy $U(N)$ k -instanton partition function [27]:

$$Z_{k,N}(\epsilon_1, \epsilon_2, \vec{a}, z) = \frac{1}{k!} \sum_{\vec{m} \in \mathbb{Z}^k} \int_{\mathbb{R}^k} \left[\prod_{a=1}^k \frac{d\sigma_a}{2\pi} z^{i\sigma_a - \frac{m_a}{2}} \bar{z}^{i\sigma_a + \frac{m_a}{2}} \right] \left[\prod_{a < b}^k \left(\frac{m_{ab}^2}{4} + \sigma_{ab}^2 \right) \right] Z_{IJ} Z_{\text{adj}}, \quad (2.2)$$

where $z = e^{-2\pi\zeta + i\theta}$ with the theta angle θ , $m_{ab} = m_a - m_b$, and $\sigma_{ab} = \sigma_a - \sigma_b$. Here

$$Z_{IJ} = \prod_{a=1}^k \prod_{\alpha=1}^N \frac{\Gamma(0^+ - i\sigma_a + ira_\alpha - \frac{1}{2}m_a)}{\Gamma(1 + i\sigma_a - ira_\alpha - \frac{1}{2}m_a)} \frac{\Gamma(0^+ + i\sigma_a - ir(a_\alpha - \epsilon) + \frac{1}{2}m_a)}{\Gamma(1 - i\sigma_a + ir(a_\alpha - \epsilon) + \frac{1}{2}m_a)} \quad (2.3)$$

is the one loop determinant of the chiral multiplets including the chiral fields I_α , J_α , and

$$Z_{\text{adj}} = \prod_{a,b=1}^k \frac{\Gamma(1 - i\sigma_{ab} - ir\epsilon - \frac{1}{2}m_{ab})}{\Gamma(i\sigma_{ab} + ir\epsilon - \frac{1}{2}m_{ab})} \frac{\Gamma(-i\sigma_{ab} + ir\epsilon_1 - \frac{1}{2}m_{ab})}{\Gamma(1 + i\sigma_{ab} - ir\epsilon_1 - \frac{1}{2}m_{ab})} \frac{\Gamma(-i\sigma_{ab} + ir\epsilon_2 - \frac{1}{2}m_{ab})}{\Gamma(1 + i\sigma_{ab} - ir\epsilon_2 - \frac{1}{2}m_{ab})} \quad (2.4)$$

is the one loop determinant of the chiral multiplets including the chiral fields χ , $B_{1,2}$, where r is the radius of S^2 , and $\epsilon = \epsilon_1 + \epsilon_2$.

In the large radius phase $\zeta \gg 0$, the above partition function can be written as [27]

$$Z_{k,N}(\epsilon_1, \epsilon_2, \vec{a}, z) = \frac{1}{k!} \oint \left[\prod_{a=1}^k \frac{d(r\lambda_a)}{2\pi i} (z\bar{z})^{-r\lambda_a} \right] Z_L |Z_V(z)|^2, \quad (2.5)$$

where

$$\begin{aligned} Z_L &= \left(\frac{\Gamma(1 - ir\epsilon)\Gamma(ir\epsilon_1)\Gamma(ir\epsilon_2)}{\Gamma(ir\epsilon)\Gamma(1 - ir\epsilon_1)\Gamma(1 - ir\epsilon_2)} \right)^k \\ &\times \prod_{a=1}^k \prod_{\alpha=1}^N \frac{\Gamma(r\lambda_a + ira_\alpha)\Gamma(0^+ - r\lambda_a - ir(a_\alpha - \epsilon))}{\Gamma(1 - r\lambda_a - ira_\alpha)\Gamma(1 + r\lambda_a + ir(a_\alpha - \epsilon))} \\ &\times \prod_{\substack{a,b=1 \\ (a \neq b)}}^k r\lambda_{ab} \frac{\Gamma(1 + r\lambda_{ab} - ir\epsilon)\Gamma(r\lambda_{ab} + ir\epsilon_1)\Gamma(r\lambda_{ab} + ir\epsilon_2)}{\Gamma(-r\lambda_{ab} + ir\epsilon)\Gamma(1 - r\lambda_{ab} - ir\epsilon_1)\Gamma(1 - r\lambda_{ab} - ir\epsilon_2)}, \end{aligned} \quad (2.6)$$

$$\begin{aligned}
 Z_V(z) = & \sum_{d_1, \dots, d_k=0}^{\infty} z^{\sum_{a=1}^k d_a} \prod_{a=1}^k \prod_{\alpha=1}^N \frac{(-r\lambda_a - ir(a_\alpha - \epsilon))_{d_a}}{(1 - r\lambda_a - ira_\alpha)_{d_a}} \\
 & \times \prod_{1 \leq a < b \leq k} \frac{d_{ba} + r\lambda_{ab}}{r\lambda_{ab}} \frac{(1 + r\lambda_{ab} - ir\epsilon)_{d_{ba}} (r\lambda_{ab} + ir\epsilon_1)_{d_{ba}} (r\lambda_{ab} + ir\epsilon_2)_{d_{ba}}}{(r\lambda_{ab} + ir\epsilon)_{d_{ba}} (1 + r\lambda_{ab} - ir\epsilon_1)_{d_{ba}} (1 + r\lambda_{ab} - ir\epsilon_2)_{d_{ba}}},
 \end{aligned} \tag{2.7}$$

$\lambda_{ab} = \lambda_a - \lambda_b$, $d_{ab} = d_a - d_b$, and $(x)_d = \frac{\Gamma(x+d)}{\Gamma(x)}$ is the Pochhammer symbol. Here by shifting the theta angle θ , we have scaled z as $z \rightarrow (-1)^{N+k-1}z$ for simplicity. The contours in (2.5) are enclosing the imaginary axes counterclockwise, and the simple poles which give residues only come from the numerators in (2.6) given by $\lambda_a = -ia_\alpha$, $\lambda_{ab} = -i\epsilon_1$, and $\lambda_{ab} = -i\epsilon_2$. As the result the simple poles are labeled by an N -tuple of Young diagrams $\vec{\mu} = (\mu_1, \dots, \mu_N)$ with $k = |\vec{\mu}| = \sum_{\alpha=1}^N |\mu_\alpha|$, where $|\mu_\alpha|$ is the number of boxes of the Young diagram μ_α .

As an example, in the $N = 2$ and $k = 3$ case, we see that the simple poles are classified by ten types of two-tuple of Young diagrams $\vec{\mu}_{1, \dots, 10}$ as

$$\begin{aligned}
 \vec{\mu}_1 = \left(\begin{array}{|c|} \hline \square \\ \hline \end{array}, \bullet \right) : & \quad \lambda_1 = -ia_1, \quad \lambda_2 = -ia_1 - i\epsilon_1, \quad \lambda_3 = -ia_1 - 2i\epsilon_1, \\
 \vec{\mu}_2 = \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \bullet \right) : & \quad \lambda_1 = -ia_1, \quad \lambda_2 = -ia_1 - i\epsilon_1, \quad \lambda_3 = -ia_1 - i\epsilon_2, \\
 \vec{\mu}_3 = \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \square \right) : & \quad \lambda_1 = -ia_1, \quad \lambda_2 = -ia_1 - i\epsilon_1, \quad \lambda_3 = -ia_2,
 \end{aligned} \tag{2.8}$$

$\vec{\mu}_4 = (\square\square, \bullet)$ obtained by $\epsilon_1 \leftrightarrow \epsilon_2$ in $\vec{\mu}_1$, $\vec{\mu}_5 = (\square, \square)$ obtained by $\epsilon_1 \leftrightarrow \epsilon_2$ in $\vec{\mu}_3$, and $\vec{\mu}_{6, \dots, 10} = (\bullet, \begin{array}{|c|} \hline \square \\ \hline \end{array}), (\bullet, \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}), (\square, \begin{array}{|c|} \hline \square \\ \hline \end{array}), (\bullet, \square\square), (\square, \square)$ obtained by $a_1 \leftrightarrow a_2$ in $\vec{\mu}_{1, \dots, 5}$.

Let us consider the degenerate limit $r \rightarrow 0$ of S^2 . Under this limit $(z\bar{z})^{-r\lambda_a} = 1 + \mathcal{O}(r)$,

$$Z_V(z) = 1 + \mathcal{O}(r^N), \tag{2.9}$$

and using $\Gamma(r) = r^{-1} + \mathcal{O}(r^0)$ to estimate Z_L , one finds that the stringy $U(N)$ k -instanton partition function (2.5) yields the $U(N)$ k -instanton (Nekrasov) partition function $Z_{k,N}^{\text{Nek}}$ [27]:

$$Z_{k,N}(\epsilon_1, \epsilon_2, \vec{a}, z) = \frac{1}{(ir)^{2kN}} Z_{k,N}^{\text{Nek}}(\epsilon_1, \epsilon_2, \vec{a}) + \text{stringy corrections}, \tag{2.10}$$

where

$$\begin{aligned}
 Z_{k,N}^{\text{Nek}}(\epsilon_1, \epsilon_2, \vec{a}) = & \frac{\epsilon^k}{k!(2\pi i\epsilon_1\epsilon_2)^k} \oint \prod_{a=1}^k \frac{d\sigma_a}{\prod_{\alpha=1}^N (\sigma_a + a_\alpha)(i0^+ + \epsilon - (\sigma_a + a_\alpha))} \\
 & \times \prod_{1 \leq a < b \leq k} \frac{\sigma_{ab}^2(\sigma_{ab}^2 - \epsilon^2)}{(\sigma_{ab}^2 - \epsilon_1^2)(\sigma_{ab}^2 - \epsilon_2^2)},
 \end{aligned} \tag{2.11}$$

and the contours are enclosing the real axes counterclockwise [6, 8, 9]. The exponent of the leading behavior r^{-2kN} coincides with $\dim_{\mathbb{C}} \mathcal{M}_{k,N} = 2kN$. It is useful to express

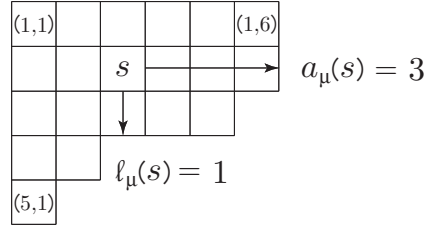


Figure 1. An example of Young diagram μ with $|\mu| = 20$. Let $s = (i, j) \in \mu$ be the label of the box at the i -th low and the j -th column of μ . At $s = (2, 3)$, $a_\mu(s) = 3$ and $\ell_\mu(s) = 1$.

the Nekrasov partition function by using N -tuple of Young diagrams $\vec{\mu} = (\mu_1, \dots, \mu_N)$ as [6, 10, 60, 61]

$$Z_{k,N}^{\text{Nek}}(\epsilon_1, \epsilon_2, \vec{a}) = \sum_{|\vec{\mu}|=k} \frac{1}{\prod_{\alpha,\beta=1}^N n_{\alpha\beta}^{\vec{\mu}}(\epsilon_1, \epsilon_2, \vec{a})}, \quad (2.12)$$

where

$$n_{\alpha\beta}^{\vec{\mu}}(\epsilon_1, \epsilon_2, \vec{a}) = \prod_{s \in \mu_\alpha} (a_{\alpha\beta} - \epsilon_1 \ell_{\mu_\beta}(s) + \epsilon_2 (a_{\mu_\alpha}(s) + 1)) \times \prod_{t \in \mu_\beta} (a_{\alpha\beta} + \epsilon_1 (\ell_{\mu_\alpha}(t) + 1) - \epsilon_2 a_{\mu_\beta}(t)). \quad (2.13)$$

Here $a_{\alpha\beta} = a_\alpha - a_\beta$, and $a_\mu(s)$ and $\ell_\mu(s)$ are the arm- and leg-length, respectively. In figure 1 we describe an example of Young diagram μ with $|\mu| = 20$. The expansion (2.10) contains the stringy corrections $\mathcal{O}(r^{-2kN+1})$ to the Nekrasov partition function, and the main subject of this paper is to study these corrections.

2.2 Stringy corrections

As discussed in [27], Z_L in (2.6) contains the perturbative α' corrections. In the supersymmetric localization [25, 26], these corrections depends on the regularization scheme of the infinite products in the one loop determinant of the chiral multiplets. In [25, 26] this ambiguity was fixed by the zeta function regularization $[\prod_{n=0}^\infty (x+n)]_{\text{reg}} = \frac{\sqrt{2\pi}}{\Gamma(x)}$ (see appendix A.1). By

$$\frac{\Gamma(-x)}{\Gamma(x)} = -\exp\left(2\gamma x + 2\sum_{s=1}^\infty \frac{\zeta(2s+1)}{2s+1} x^{2s+1}\right), \quad (2.14)$$

we see that the perturbative α' corrections in (2.10) contain the Euler constant γ and the simple zeta values $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$. On the other hand, $Z_V(z)$ in (2.7) contains the non-perturbative α' corrections and does not depend on the regularization scheme. Especially it was proposed that $Z_V(z)$ gives the Givental's \mathcal{I} -function [45, 46] of the ADHM moduli space $\mathcal{M}_{k,N}$ [27, 62]:

$$\mathcal{I}(z) = Z_V(z). \quad (2.15)$$

Here $\lambda_1, \dots, \lambda_k$ are identified with the Chern roots of the tautological bundle on $\mathcal{M}_{k,N}$ (for $N = 1$, see [63]). The inverse radius r^{-1} of the two sphere S^2 is identified with

the equivariant parameter which gives the generator of S^1 acting on the S^2 . By the equivariant mirror map, the \mathcal{I} -function is related with the small \mathcal{J} -function $\mathcal{J}(z)$. Then by expanding the \mathcal{J} -function around $r = 0$, one can obtain the T^{N+2} -equivariant Gromov-Witten invariants.⁶ It is known that the coefficient of r^1 in the expansion around $r = 0$ of the \mathcal{I} -function gives the equivariant mirror map. As discussed in [27], by (2.15) and the behavior (2.9) of $Z_V(z)$, one finds that the equivariant mirror maps for the $N \geq 2$ cases are trivial: $\mathcal{J}(z) = \mathcal{I}(z) = Z_V(z)$, but for the $N = 1$ case the equivariant mirror map is needed as [63]

$$\mathcal{J}(z) = (1 - z)^{ikr\epsilon} \mathcal{I}(z) = (1 - z)^{ikr\epsilon} Z_V(z). \quad (2.16)$$

In (2.5), by taking the leading terms of $Z_L \sim \mathcal{O}(r^{-2kN})$ and $Z_V(z) \sim 1$ around $r = 0$, we obtain the stringy instanton partition function without the quantum stringy (α') corrections:

$$Z_{k,N}^{\text{classic}}(\epsilon_1, \epsilon_2, \vec{a}, z) = \frac{1}{(ir)^{2kN}} \sum_{|\vec{\mu}|=k} \frac{(z\bar{z})^{ir \sum_{\alpha=1}^N (a_\alpha |\mu_\alpha| + \epsilon_1 n(\mu_\alpha) + \epsilon_2 n(\mu_\alpha^t))}}{\prod_{\alpha,\beta=1}^N n_{\alpha\beta}^{\vec{\mu}}(\epsilon_1, \epsilon_2, \vec{a})}, \quad (2.17)$$

where $n(\mu) = \sum_{s \in \mu} \ell_\mu(s)$ and $n(\mu^t) = \sum_{s \in \mu} a_\mu(s)$. Especially in the anti-self-dual case $\epsilon_1 = -\epsilon_2 = \hbar$, one finds that the stringy instanton partition function does not have the α' corrections [27] (see also [47]), and thus

$$Z_{k,N}(\hbar, -\hbar, \vec{a}, z) = Z_{k,N}^{\text{classic}}(\hbar, -\hbar, \vec{a}, z). \quad (2.18)$$

In section 3 and section 4 we study the classical and quantum stringy corrections, respectively.

3 Classical stringy corrections

In this section we study the classical stringy corrections. Let us define the generating function of the classical stringy $U(N)$ instanton partition functions (2.17) by

$$Z_N^{\text{cSI}}(\epsilon_1, \epsilon_2, \vec{a}, z, \Lambda) = 1 + \sum_{k=1}^{\infty} (r\Lambda)^{2kN} Z_{k,N}^{\text{classic}}(\epsilon_1, \epsilon_2, \vec{a}, z), \quad (3.1)$$

where Λ is the dynamical scale in four dimensional gauge theory. This partition function gives the instanton (“non-perturbative” in the gauge theoretic sense) partition function without the α' corrections of the N D5-world volume theory on $\mathbb{C}^2 \times \mathbb{P}^1$ that was dimensionally reduced to the \mathbb{C}^2 . In [27], the “perturbative” (in the gauge theoretic sense) partition function

$$Z_N^{\text{D5}}(\epsilon_1, \epsilon_2, \vec{a}) = Z_N^{\text{cD5}}(\epsilon_1, \epsilon_2, \vec{a}) \times Z_N^{\text{qD5}}(\epsilon_1, \epsilon_2, \vec{a}) \quad (3.2)$$

⁶The $N + 2$ dimensional torus $T^{N+2} = U(1)^N \times U(1)^2$ acts on $\mathcal{M}_{k,N}$ as $U(1)^N$: $(B_1, B_2, I, J) \mapsto (B_1, B_2, IT_a^{-1}, T_a J)$ where $T_a = \text{diag}(e^{ia_1}, \dots, e^{ia_N})$, and $U(1)^2$: $(B_1, B_2, I, J) \mapsto (T_{\epsilon_1} B_1, T_{\epsilon_2} B_2, I, T_{\epsilon_1} T_{\epsilon_2} J)$ where $T_{\epsilon_{1,2}} = e^{i\epsilon_{1,2}}$. This action introduces the equivariant parameters a_α , $\alpha = 1, \dots, N$ and $\epsilon_{1,2}$ corresponding to the twisted masses in the GLSM as described in table 1.

was also computed. Here

$$Z_N^{\text{cD5}}(\epsilon_1, \epsilon_2, \vec{a}) = \prod_{\substack{\alpha, \beta=1 \\ (\alpha \neq \beta)}}^N \Gamma_2(a_{\alpha\beta} | \epsilon_1, \epsilon_2) \quad (3.3)$$

gives the well-known perturbative partition function of the four dimensional $\mathcal{N} = 2$ $U(N)$ gauge theory on the Omega background [11, 64] (see also appendix A.2). The quantum stringy part

$$Z_N^{\text{qD5}}(\epsilon_1, \epsilon_2, \vec{a}) = \prod_{\substack{\alpha, \beta=1 \\ (\alpha \neq \beta)}}^N \frac{\Gamma_3(a_{\alpha\beta} | \epsilon_1, \epsilon_2, \frac{1}{ir})}{\Gamma_3(a_{\alpha\beta} | \epsilon_1, \epsilon_2, -\frac{1}{ir})} \quad (3.4)$$

contains the perturbative α' corrections, where $\Gamma_r(z | \omega_1, \dots, \omega_r)$ is the multiple gamma function (see appendix A), and one finds $Z_N^{\text{qD5}}(\hbar, -\hbar, \vec{a}) = 1$. Then in the anti-self-dual case $\epsilon_1 = -\epsilon_2 = \hbar$, there are no α' corrections as in the instanton partition function (2.18),

$$Z_N^{\text{D5}}(\hbar, -\hbar, \vec{a}) = Z_N^{\text{cD5}}(\hbar, -\hbar, \vec{a}). \quad (3.5)$$

In the following we show that the classical stringy partition functions (3.1) (and (3.3)) are reduced from a four dimensional limit of the K-theoretic Nekrasov partition function with five dimensional Chern-Simons term. We also find that the instantons with the classical stringy corrections can be embedded into (refined) topological string theory.

3.1 Relation with K-theoretic instanton partition function

The K-theoretic $U(N)$ instanton (Nekrasov) partition function with five dimensional Chern-Simons term $\int \text{Tr} A \wedge F \wedge F$ [65] is given by [39, 40]

$$Z_{N,m}^{\text{K-Nek}}(\epsilon_1, \epsilon_2, \vec{a}, \Lambda) = \sum_{\vec{\mu}} \frac{(e^{-\frac{1}{2}(N-m)\epsilon} \Lambda^{2N})^{|\vec{\mu}|}}{\prod_{\alpha, \beta=1}^N N_{\alpha\beta}^{\vec{\mu}}(\epsilon_1, \epsilon_2, \vec{a})} \cdot e^{m \sum_{\alpha=1}^N (a_{\alpha} |\mu_{\alpha}| + \epsilon_1 n(\mu_{\alpha}) + \epsilon_2 n(\mu_{\alpha}^t))}, \quad (3.6)$$

where $m \in \mathbb{Z}$ is the Chern-Simons coefficient, and

$$N_{\alpha\beta}^{\vec{\mu}}(\epsilon_1, \epsilon_2, \vec{a}) = \prod_{s \in \mu_{\alpha}} \left(1 - Q_{\alpha\beta} t^{-\ell_{\mu_{\beta}}(s)} q^{-a_{\mu_{\alpha}}(s)-1} \right) \prod_{t \in \mu_{\beta}} \left(1 - Q_{\alpha\beta} t^{\ell_{\mu_{\alpha}}(t)+1} q^{a_{\mu_{\beta}}(t)} \right), \quad (3.7)$$

$t = e^{-\epsilon_1}$, $q = e^{\epsilon_2}$, $Q_{\alpha\beta} = e^{-a_{\alpha\beta}}$. Let β be the radius of the five dimensional circle. After scaling $a_{\alpha} \rightarrow i\beta a_{\alpha}$, $\epsilon_{1,2} \rightarrow i\beta \epsilon_{1,2}$, $\Lambda \rightarrow i e^{-\frac{im}{4N}\beta\epsilon} \beta\Lambda$, by taking a four dimensional limit

$$\beta \rightarrow 0, \quad m \rightarrow \infty \quad \text{with fixed} \quad \beta m = r \log z \bar{z} = -4\pi\zeta r, \quad (3.8)$$

we see that the K-theoretic Nekrasov partition function yields the classical stringy instanton partition function (3.1):

$$Z_{N,m}^{\text{K-Nek}}(\epsilon_1, \epsilon_2, \vec{a}, \Lambda) \longrightarrow Z_N^{\text{cSI}}(\epsilon_1, \epsilon_2, \vec{a}, z, \Lambda). \quad (3.9)$$

The four dimensional limit (3.8) relates (the strong coupling limit of) the Chern-Simons coefficient m to the Kähler modulus ζ . The K-theoretic “perturbative” (in the gauge

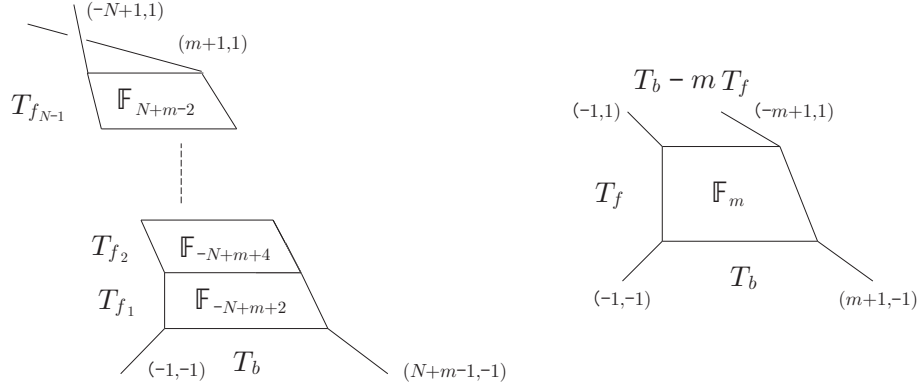


Figure 2. The left figure describes the toric (web) diagram of the $SU(N)$ geometry $X_{N,m}$ labeled by $m \in \mathbb{Z}$. It contains $N - 1$ Hirzebruch surfaces $\{\mathbb{F}_{-N+m+2\ell}\}_{\ell=1}^{N-1}$ as the compact divisors. The right figure describes the Hirzebruch surface \mathbb{F}_m with two Kähler moduli T_f and T_b .

theoretic sense) partition function $Z_N^{\text{K-pert}}$ [11, 40, 66] does not depend on the Chern-Simons term, and yields the perturbative partition function Z_N^{cD5} in (3.3) without the α' corrections. Note that in the anti-self-dual case $\epsilon_1 = -\epsilon_2 = \hbar$, as in (2.18) and (3.5) the stringy partition functions do not receive the α' corrections, and the limit (3.8) of the K-theoretic perturbative/non-perturbative (instanton) partition functions completely coincide with the stringy partition functions.

3.2 Relation with topological strings

By geometric engineering [3–5], it is known that the K-theoretic $SU(N)$ Nekrasov partition function with a Chern-Simons coefficient m coincides with the partition function $Z_{N,m}^{\text{refA}}$ of the refined topological A-model on a local toric Calabi-Yau threefold $X_{N,m}$ given by ALE space fibration of A_{N-1} type over \mathbb{P}^1 [22–24] (see [13–18] for the unrefined case). Topological type of this $SU(N)$ geometry $X_{N,m}$ described in figure 2 is labeled by an integer m which is identified with the Chern-Simons coefficient [39]. The $SU(N)$ geometry $X_{N,m}$ has one modulus T_b of the base \mathbb{P}^1 and $N - 1$ moduli T_{f_α} , $\alpha = 1, \dots, N - 1$ of the fiber consisting of $N - 1$ resolved \mathbb{P}^1 's. Here T_b is identified with the dynamical scale Λ , and T_{f_α} are identified with $N - 1$ independent Coulomb moduli a_α in the $SU(N)$ gauge theory [4]:

$$e^{-T_b} \sim (\beta\Lambda)^{2N}, \quad T_{f_\alpha} \sim \beta a_{\alpha\alpha+1} = \beta(a_\alpha - a_{\alpha+1}). \quad (3.10)$$

Then by taking the limit (3.8), we find that the partition function $Z_{N,m}^{\text{refA}}$ of the refined A-model on the family of the local toric Calabi-Yau threefolds $X_{N,m}$ labeled by m yields the classical stringy partition function of the four dimensional $SU(N)$ gauge theory:

$$Z_{N,m}^{\text{refA}}(\epsilon_1, \epsilon_2, T_{f_\alpha}, T_b) \longrightarrow Z_N^{\text{cD5}}(\epsilon_1, \epsilon_2, \vec{a}) \times Z_N^{\text{cSI}}(\epsilon_1, \epsilon_2, \vec{a}, z, \Lambda) \Big|_{\sum_{\alpha=1}^N a_\alpha = 0}. \quad (3.11)$$

The anti-self-dual case $\epsilon_1 = -\epsilon_2 = \hbar$ corresponds to the topological string (unrefined) limit, and \hbar is identified with the topological string coupling constant g_s . Therefore the

(unrefined) A-model partition function under the four dimensional limit (3.8) completely coincides with the stringy partition function on the anti-self-dual Omega background.⁷

3.3 Brane construction and geometric engineering

In this section by revisiting a correspondence discussed in [41] between the world volume theory of N D5-branes on $\mathbb{C}^2 \times \{\text{vanishing } \mathbb{P}^1\}$ and the geometrically engineered quantum field theory by local Calabi-Yau threefold with resolved A_{N-1} singularity, we give a physical explanation of the relations (3.9) and (3.11). As described in section 2, the intersecting D1- and D5-branes in type IIB string theory on $\mathbb{C}^2 \times T^*\mathbb{P}^1 \times \mathbb{C}$ lead to the instantons with the stringy corrections. In the D5-world volume theory, after the compactification of the \mathbb{P}^1 , to preserve eight supercharges in the four dimensions, the $U(1)$ spin connection on the \mathbb{P}^1 needs to be embedded into the $SO(4)$ R-symmetry [54]. This topological twist breaks the R-symmetry as $SO(4) \cong SU(2) \times SU(2)$ to $U(1) \times SU(2)$, and one obtains six dimensional $\mathcal{N} = (1, 0)$ $U(N)$ supersymmetric gauge theory on $\mathbb{C}^2 \times \mathbb{P}^1$ which leads to four dimensional $\mathcal{N} = 2$ supersymmetric gauge theory.

Type IIB theory \rightarrow Type IIB theory: firstly, by taking the S-duality in type IIB string theory, the N D5- and k D1-branes are turned into NS5-branes and the fundamental strings wrapping around the compactified \mathbb{P}^1 .

Type IIB theory \rightarrow Type IIA theory: as was shown in [67], the N NS5-branes in type IIB (IIA) string theory is equivalent to type IIA (IIB) string theory on an A_{N-1} ALE space. By this duality, we obtain type IIA string theory compactified on a local Calabi-Yau threefold given by ALE space fibration of A_{N-1} type over \mathbb{P}^1 , and then four dimensional $\mathcal{N} = 2$ $SU(N)$ supersymmetric gauge theory is geometrically engineered [3–5]. The low energy type IIA supergravity theory has the Chern-Simons term $\int B^{(2)} \wedge dC^{(3)} \wedge dC^{(3)}$, where $B^{(2)}$ is the NS-NS B field and $C^{(3)}$ is the R-R 3-form field. From this Chern-Simons term, by integration on the local Calabi-Yau threefold, we obtain

$$\zeta_b g_{\alpha\beta\gamma} \int \phi_\alpha dA_\beta \wedge dA_\gamma. \quad (3.12)$$

Here $\phi_\alpha = \frac{1}{\zeta_b} \int_{\mathcal{C}_\alpha} B^{(2)}$ and $A_\alpha = \int_{\mathcal{C}_\alpha} C^{(3)}$ defined for the two cycles \mathcal{C}_α , $\alpha = 1, \dots, N-1$ on the ALE space give scalar fields and gauge fields on \mathbb{C}^2 , respectively, and $g_{\alpha\beta\gamma}$ is given by a classical triple intersection number of divisors on the local Calabi-Yau threefold.⁸ It is known that the NS-NS B field introduces a noncommutative parameter in the four dimensions, and it also introduces the FI parameter in the ADHM moduli space [69–71]. Therefore we argue that the coefficient $\zeta_b g_{\alpha\beta\gamma}$ is identified with the FI parameter.

Let $X_N(\zeta)$ be the local Calabi-Yau threefold obtained from the toric Calabi-Yau threefold $X_{N,m}$ in figure 2 at the phase under the limit (3.8).⁹ By considering this Calabi-Yau

⁷A similar limit to (3.8) was previously discussed in [48] to relate the simple Hurwitz theory with the topological A-model on the local toric Calabi-Yau threefold $X_{1,m} = \mathcal{O}(m-2) \oplus \mathcal{O}(-m) \rightarrow \mathbb{P}^1$. In appendix B, we discuss and summarize such relations.

⁸When the two cycles \mathcal{C}_α shrink to points, the $U(1)^{N-1}$ gauge symmetry is enhanced to $SU(N)$ [68].

⁹The parameter m in $X_{N,m}$ gives a classical triple intersection number.

threefold $X_N(\zeta)$, as discussed above, we find an interaction term like (3.12). As discussed in [39], the parameter ζ obtained by the scaling of the intersection number m in $X_{N,m}$ is related with the coefficient of (3.12), and so identified with the FI parameter. Then we see that the instanton partition function with the term (3.12) on the Omega background is given by the classical stringy instanton partition functions (2.17):

$$\int_{\mathcal{M}_{k,N}} e^{i\zeta\phi} \Big|_{\sum_{\alpha=1}^N a_\alpha=0} = Z_{k,N}^{\text{classic}}(\epsilon_1, \epsilon_2, \vec{a}, z) \Big|_{\sum_{\alpha=1}^N a_\alpha=0}, \quad (3.13)$$

where $e^{i\zeta\phi}$ is an element of the $T = \text{U}(1)^N \times \text{U}(1)^2$ -equivariant cohomology $H_T^*(\mathcal{M}_{k,N})$ [8]. Note that we also have the fundamental strings wrapping around the base \mathbb{P}^1 which give the degree k worldsheet instanton for this base whose Kähler modulus gives the dynamical scale Λ .

Type IIA theory \rightarrow M-theory: lifting to the M-theory leads to the relation (3.9) between the K-theoretic $\text{SU}(N)$ instanton partition function with five dimensional Chern-Simons term and the classical stringy $\text{SU}(N)$ instanton partition function in four dimensions. We see that by this five dimensional lift, the FI parameter ζ is quantized, and it gives a five dimensional Chern-Simons coefficient $m \in \mathbb{Z}$. As the result, we argue that the K-theoretic classical stringy instanton partition function is given by the K-theoretic instanton partition function (3.6) with five dimensional Chern-Simons coefficient.

The K-theoretic instanton partition function and the refined topological A-model partition function do not capture the quantum stringy corrections. In the next section, we study these quantum corrections.

4 Quantum stringy corrections

In the general Omega background, the stringy instanton partition function (2.5) has the quantum stringy (α') corrections, and it gives the quantum-corrected Kähler potential $K_{k,N}$ on the Kähler moduli space of the ADHM moduli space $\mathcal{M}_{k,N}$ [27, 42, 43, 62]:

$$e^{-K_{k,N}(\epsilon_1, \epsilon_2, \vec{a}, z)} = Z_{k,N}(\epsilon_1, \epsilon_2, \vec{a}, z) \quad (4.1)$$

up to an ambiguity by Kähler transformations $K_{k,N}(z, \bar{z}) \rightarrow K_{k,N}(z, \bar{z}) + f(z) + \overline{f(\bar{z})}$ where $f(z)$ is a holomorphic function of z . In the following, let us fix this ambiguity by a normalization

$$Z_{k,N}^{\text{norm}}(\epsilon_1, \epsilon_2, \vec{a}, z) = (z\bar{z})^{-ir \frac{k}{N} \sum_{\alpha=1}^N a_\alpha} \left(\frac{\Gamma(-ir\epsilon_1)\Gamma(-ir\epsilon_2)}{\Gamma(ir\epsilon_1)\Gamma(ir\epsilon_2)} \right)^{kN} Z_{k,N}(\epsilon_1, \epsilon_2, \vec{a}, z). \quad (4.2)$$

Here the first normalization factor $(z\bar{z})^{-ir \frac{k}{N} \sum_{\alpha=1}^N a_\alpha}$ shifts the classical stringy corrections, and then we see that this normalized partition function is invariant under simultaneous constant shift $a_\alpha \rightarrow a_\alpha + c$, $\alpha = 1, \dots, N$. As the result, the $\text{U}(1)$ factor is decoupled from the partition function: $\text{U}(N) \rightarrow \text{SU}(N)$. By (2.14), the second normalization factor removes the dependence of the Euler constant γ [27], and fixes the precoefficients of the

simple zeta values $\zeta(s)$ which receive the perturbative α' corrections. These normalizations fix the regularization scheme in the perturbative α' corrections mentioned in section 2.2. In this section, we study the structure of this partition function for $N = 1, 2, 3$ cases, and find a universal structure of the quantum stringy corrections for arbitrary instanton charge k as in (4.4), (4.28), and (4.55).

4.1 Stringy U(1) instanton counting

In the $N = 1$ case, the ADHM moduli space $\mathcal{M}_{k,1}$ is isomorphic to the Hilbert scheme of points $\text{Hilb}_k(\mathbb{C}^2)$ on \mathbb{C}^2 . In this case, an equivariant mirror map described in (2.16) is needed for reproducing the correct Gromov-Witten invariants of $\text{Hilb}_k(\mathbb{C}^2)$, and thus we study [27]

$$\widehat{Z}_{k,1}^{\text{norm}}(\epsilon_1, \epsilon_2, z) = (1+z)^{ikr\epsilon} (1+\bar{z})^{ikr\epsilon} Z_{k,1}^{\text{norm}}(\epsilon_1, \epsilon_2, -z). \quad (4.3)$$

Note that the Coulomb modulus a has been removed by the first normalization factor in (4.2). We find that its asymptotic expansion around $r = 0$ is given by

$$\begin{aligned} \widehat{Z}_{k,1}^{\text{norm}}(\epsilon_1, \epsilon_2, z) = & \sum_{\ell=0}^{\infty} \frac{(-1)^\ell G_{k,\ell}}{(\epsilon_1 \epsilon_2)^{k-\ell}} l^{2\ell}(z) \\ & - \frac{G_{k-2,0}\epsilon}{(\epsilon_1 \epsilon_2)^{k-1}} \left[e^{-K_1^{(3)}(z)} + \epsilon e^{-K_1^{(4)}(z)} + \epsilon^2 e^{-K_1^{(5)}(z)} + \epsilon^3 e^{-K_1^{(6)}(z)} \right] \\ & + \frac{9G_{k-3,0}\epsilon}{(\epsilon_1 \epsilon_2)^{k-2}} \left[e^{-K_{1;1}^{(5)}(z)} + \epsilon e^{-K_{1;1}^{(6)}(z)} \right] + \frac{2G_{k-4,0}\epsilon}{(\epsilon_1 \epsilon_2)^{k-2}} \left[e^{-K_{1;2}^{(5)}(z)} + \epsilon e^{-K_{1;2}^{(6)}(z)} \right] \\ & + \frac{2G_{k,2}\epsilon}{(\epsilon_1 \epsilon_2)^{k-2}} \left[e^{-K_{1;3}^{(5)}(z)} + \epsilon e^{-K_{1;3}^{(6)}(z)} \right] + \epsilon \times \mathcal{O}(r^{-2k+7}), \end{aligned} \quad (4.4)$$

where $\epsilon_{1,2} = ir\epsilon_{1,2}$, $\epsilon = \epsilon_1 + \epsilon_2 = ir\epsilon$, and

$$l^n(z) \equiv \frac{1}{n!} \log^n z \bar{z}. \quad (4.5)$$

The coefficients $e^{-K_1^{(n)}(z)}$, $n = 3, \dots, 6$, and $e^{-K_{1;i}^{(n)}(z)}$, $n = 5, 6, i = 1, 2, 3$ are given below. We have checked this expansion up to $k = 5$ (-instanton), and for the classical stringy corrections up to $k = 7$ (-instanton). Here

$$G_{k \geq 0, \ell} = \sum_{|\mu|=k} \frac{1}{\prod_{s \in \mu} h_\mu(s)^2} \left(\sum_{s \in \mu} c_\mu(s) \right)^{2\ell}, \quad G_{k < 0, \ell} \equiv 0, \quad (4.6)$$

where $c_\mu(s) = \sum_{s \in \mu} (a_\mu(s) - \ell_\mu(s)) = \sum_{(i,j) \in \mu} (j - i)$, and $h_\mu(s) = a_\mu(s) + \ell_\mu(s) + 1$ is the hook length. Some of these numbers are given by

$$\begin{aligned} G_{k,0} &= \frac{1}{k!}, & G_{k,1} &= \frac{k(k-1)}{2k!}, & G_{k,2} &= \frac{k(k-1)(3k^2+k-12)}{2^2 k!}, \\ G_{k,3} &= \frac{k(k-1)(15k^4+30k^3-105k^2-700k+1344)}{2^3 k!}. \end{aligned} \quad (4.7)$$

One finds that $G_{k,\ell}$ coincides with the equivariant classical intersection number $\langle D^{2\ell} \rangle^{\text{cl}}|_{\epsilon=0}$ of the divisor class D in the T -equivariant cohomology $H_T^*(\text{Hilb}_k(\mathbb{C}^2), \mathbb{Q})$ computed

in (C.8), where $T = \text{U}(1)^2$ acts on \mathbb{C}^2 . As discussed in appendix B, these numbers also give the disconnected simple Hurwitz numbers of \mathbb{P}^1 which count degree k covers of \mathbb{P}^1 only with simple branch points (see (B.4)).

The normalized stringy $\text{U}(1)$ instanton partition function $\hat{Z}_{k,1}^{\text{norm}}$ gives the Kähler potential on the quantum Kähler moduli space of $\text{Hilb}_k(\mathbb{C}^2)$, and by (2.15) the non-perturbative α' corrections should be given by the equivariant Gromov-Witten invariants. In the following, we explicitly describe the expansion coefficients of (4.4), and confirm that such corrections are given by the equivariant Gromov-Witten invariants computed in appendix C.

The coefficients $e^{-K_1^{(n)}(z)}$, $n = 3, \dots, 6$ are given by

$$e^{-K_1^{(3)}(z)} = \frac{1}{2}l^3(z) - 3\zeta_3 + (L_2(z) + \text{c.c.})l^1(z) - 2(L_3(z) + \text{c.c.}), \quad (4.8)$$

$$e^{-K_1^{(4)}(z)} = \frac{1}{2}l^4(z) - 2\zeta_3 l^1(z) + (L_2(z) + \text{c.c.})l^2(z) - (\tilde{L}_3(z) + \text{c.c.})l^1(z) - (L_2(z) - \text{c.c.})^2, \quad (4.9)$$

$$\begin{aligned} e^{-K_1^{(5)}(z)} = & \frac{1}{2}l^5(z) - 2\zeta_3 l^2(z) - 4\zeta_3(L_2(z) + \text{c.c.}) - 7\zeta_5 \\ & + (L_2(z) + \text{c.c.})l^3(z) - (\tilde{L}_3(z) + \text{c.c.})l^2(z) \\ & + \left[\left(\int \frac{dz}{z} \tilde{L}_3(z) + \frac{2}{3} \int \frac{dz}{z} L_1(z)^3 + \text{c.c.} \right) + 2L_2(z)L_2(\bar{z}) \right] l^1(z) \\ & - 2 \left[\left(\int \int \frac{dz^2}{z^2} \tilde{L}_3(z) + \frac{2}{3} \int \int \frac{dz^2}{z^2} L_1(z)^3 - \int \frac{dz}{z} L_2(z)^2 \right. \right. \\ & \quad \left. \left. + 2 \int \frac{dz}{z} L_1(z)^2 L_2(z) + \text{c.c.} \right) + (L_2(z) + \text{c.c.})(\tilde{L}_3(z) + \text{c.c.}) \right], \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} e^{-K_1^{(6)}(z)} = & \frac{1}{2}l^6(z) - 2\zeta_3 l^3(z) - 4\zeta_3(L_2(z) + \text{c.c.})l^1(z) + 4\zeta_3(\tilde{L}_3(z) + \text{c.c.}) \\ & + 4\zeta_3^2 - 6\zeta_5 l^1(z) + (L_2(z) + \text{c.c.})l^4(z) - (\tilde{L}_3(z) + \text{c.c.})l^3(z) \\ & + \left[\left(\int \frac{dz}{z} \tilde{L}_3(z) + \frac{2}{3} \int \frac{dz}{z} L_1(z)^3 + \text{c.c.} \right) + 2L_2(z)L_2(\bar{z}) \right] l^2(z) \\ & - \left[\int \int \frac{dz^2}{z^2} \tilde{L}_3(z) + \frac{2}{3} \int \int \frac{dz^2}{z^2} L_1(z)^3 - \frac{1}{3} \int \frac{dz}{z} L_1(z)^4 \right. \\ & \quad \left. + 2L_2(z)\tilde{L}_3(\bar{z}) + \text{c.c.} \right] l^1(z) + (\tilde{L}_3(z) + \text{c.c.})^2 \\ & - 2(L_2(z) - \text{c.c.}) \left(\int \frac{dz}{z} \tilde{L}_3(z) + \frac{2}{3} \int \frac{dz}{z} L_1(z)^3 - \text{c.c.} \right), \end{aligned} \quad (4.11)$$

where

$$\tilde{L}_3(z) \equiv L_3(z) - \int \frac{dz}{z} L_1(z)^2. \quad (4.12)$$

Note that in the perturbative α' corrections which depend on the regularization scheme, the simple zeta values $\zeta_s \equiv \zeta(s)$ are involved. We see that the non-perturbative α' corrections

are completely given in terms of the polylogarithms

$$L_0(z) \equiv \text{Li}_0(z) = \frac{z}{1-z}, \quad L_{n+1}(z) \equiv \text{Li}_{n+1}(z) = \int \frac{dz}{z} \text{Li}_n(z). \quad (4.13)$$

Especially $K_1^{(3)}(z)$ and $K_1^{(4)}(z)$ have the forms of the Kähler potentials on the Kähler moduli spaces of Calabi-Yau threefolds (E.1) and of Calabi-Yau fourfolds (E.4), respectively. Combining with the precoefficients in (4.4) we argue that the non-perturbative α' corrections in $K_1^{(n)}(z)$ are determined by the equivariant three-point function on $\text{Hilb}_k(\mathbb{C}^2)$ computed in (C.10),

$$K_1^{(n)}(z) : \langle \epsilon^{n-3} D, D, D \rangle = -\frac{k(k-1)\epsilon^{n-2}}{k!(\epsilon_1\epsilon_2)^{k-1}} \left(\frac{1}{2} + L_0(z) \right), \quad n = 3, \dots, 6, \quad (4.14)$$

where D is the observable corresponding to the Poincaré dual of the divisor class in $H_T^*(\text{Hilb}_k(\mathbb{C}^2), \mathbb{Q})$.

The coefficients $e^{-K_{1;1}^{(5)}(z)}$ and $e^{-K_{1;2}^{(5)}(z)}$ are given by

$$e^{-K_{1;1}^{(5)}(z)} = l^5(z) + \frac{16}{9}\zeta_3 l^2(z) + \frac{320}{27}\zeta_5 + (M_3(z) + \text{c.c.})l^2(z) - 3(M_4(z) + \text{c.c.})l^1(z) + 6(M_5(z) + \text{c.c.}), \quad (4.15)$$

$$e^{-K_{1;2}^{(5)}(z)} = \frac{1}{2}l^5(z) + \frac{3}{2}\zeta_3 l^2(z) + \frac{45}{4}\zeta_5 + (L_3(z) + \text{c.c.})l^2(z) - 3(L_4(z) + \text{c.c.})l^1(z) + 6(L_5(z) + \text{c.c.}). \quad (4.16)$$

The non-perturbative α' corrections in these coefficients are given in terms of

$$M_0(z) \equiv \frac{z(1-2z)}{1-z+z^2}, \quad M_{n+1}(z) \equiv \int \frac{dz}{z} M_n(z) \quad (4.17)$$

for $K_{1;1}^{(5)}(z)$, and $L_n(z)$ for $K_{1;2}^{(5)}(z)$. With the precoefficients in (4.4) we argue that these quantum corrections are determined by the equivariant three-point functions computed in (C.11) for $K_{1;1}^{(5)}(z)$, and (C.12) for $K_{1;2}^{(5)}(z)$,

$$K_{1;1}^{(5)}(z) : \langle D, Y_2^{(1)}, Y_2^{(1)} \rangle = \frac{k(k-1)(k-2)\epsilon}{k!(\epsilon_1\epsilon_2)^{k-2}} \left(1 + M_0(z) \right), \quad (4.18)$$

$$K_{1;2}^{(5)}(z) : \langle D, Y_2^{(2)}, Y_2^{(2)} \rangle = \frac{k(k-1)(k-2)(k-3)\epsilon}{2k!(\epsilon_1\epsilon_2)^{k-2}} \left(\frac{1}{2} + L_0(z) \right). \quad (4.19)$$

Here $Y_2^{(1)}$ and $Y_2^{(2)}$ are the two independent observables corresponding to the Poincaré dual of (complex) codimension two cycle classes in $H_T^*(\text{Hilb}_k(\mathbb{C}^2), \mathbb{Q})$.

The coefficients $e^{-K_{1;1}^{(6)}(z)}$ and $e^{-K_{1;2}^{(6)}(z)}$ are given by

$$e^{-K_{1;1}^{(6)}(z)} = 5l^6(z) + \frac{10}{9}\zeta_3 l^3(z) + \frac{32}{9}\zeta_3 (L_2(z) + \text{c.c.})l^1(z) - \frac{2}{9}\zeta_3 (32L_3(z) + 3M_3(z) + \text{c.c.}) - \frac{160}{27}\zeta_3^2 + \frac{106}{9}\zeta_5 l^1(z) + 2(L_2(z) + \text{c.c.})l^4(z) - 4(L_3(z) - M_3(z) + \text{c.c.})l^3(z)$$

$$\begin{aligned}
 & + 2 \left[3L_4(z) - 4M_4(z) - \int \int \frac{dz^2}{z^2} L_1(z) M_1(z) + \frac{3}{4} \int \int \frac{dz^2}{z^2} M_1(z)^2 \right. \\
 & \quad \left. + \int \frac{dz}{z} L_1(z) M_2(z) + \text{c.c.} \right] l^2(z) \\
 & - 2 \left[\left(3L_5(z) - 4M_5(z) - \int \int \int \frac{dz^3}{z^3} L_1(z) M_1(z) + \frac{3}{4} \int \int \int \frac{dz^3}{z^3} M_1(z)^2 \right. \right. \\
 & \quad \left. + \int \int \frac{dz^2}{z^2} L_1(z) M_2(z) + 3 \int \frac{dz}{z} L_1(z) M_3(z) + \frac{3}{4} \int \frac{dz}{z} M_2(z)^2 + \text{c.c.} \right) \\
 & \quad \left. - (L_2(z) + \text{c.c.})(M_3(z) + \text{c.c.}) \right] l^1(z) \\
 & - \frac{1}{2} (M_3(z) + \text{c.c.})(8L_3(z) - 3M_3(z) + \text{c.c.}) + 6(L_2(z) - \text{c.c.})(M_4(z) - \text{c.c.}), \\
 & \hspace{15em} (4.20)
 \end{aligned}$$

and

$$\begin{aligned}
 e^{-K_{1;2}^{(6)}(z)} & = 2l^6(z) + \frac{3}{2} \zeta_3 l^3(z) + 3\zeta_3 (L_2(z) + \text{c.c.}) l^1(z) - 6\zeta_3 (L_3(z) + \text{c.c.}) - \frac{9}{2} \zeta_3^2 \\
 & \quad + 12\zeta_5 l^1(z) + (L_2(z) + \text{c.c.}) l^4(z) + (L_3(z) + \text{c.c.}) l^3(z) \\
 & \quad - (3L_4(z) - L_2(z)^2 + \text{c.c.}) l^2(z) \\
 & \quad + \left[3L_5(z) + 3 \int \frac{dz}{z} L_2(z)^2 - 2L_2(z)(2L_3(z) - L_3(\bar{z})) + \text{c.c.} \right] l^1(z) \\
 & \quad - 2(L_3(z) + \text{c.c.})^2 + 6(L_2(z) - \text{c.c.})(L_4(z) - \text{c.c.}). \\
 & \hspace{15em} (4.21)
 \end{aligned}$$

We argue that the non-perturbative α' corrections in these coefficients are determined by the equivariant three-point functions as

$$K_{1;1}^{(6)}(z) : \langle \epsilon_1 \epsilon_2 \epsilon D, D, D \rangle \quad \text{and} \quad \langle \epsilon D, Y_2^{(1)}, Y_2^{(1)} \rangle, \quad (4.22)$$

$$K_{1;2}^{(6)}(z) : \langle \epsilon_1 \epsilon_2 \epsilon D, D, D \rangle \quad \text{and} \quad \langle \epsilon D, Y_2^{(2)}, Y_2^{(2)} \rangle. \quad (4.23)$$

We find that the coefficients $K_{1;3}^{(5)}(z)$ and $K_{1;3}^{(6)}(z)$ are written in terms of the other coefficients as

$$e^{-K_{1;3}^{(5)}(z)} = \frac{1}{3} \left(e^{-K_1^{(3)}(z)} l^2(z) - 4e^{-K_{1;2}^{(5)}(z)} \right) \sim l^5(z), \quad (4.24)$$

$$e^{-K_{1;3}^{(6)}(z)} = \frac{1}{3} \left(e^{-2K_1^{(3)}(z)} + e^{-K_1^{(4)}(z)} l^2(z) - 4e^{-K_{1;2}^{(6)}(z)} \right) \sim \frac{3}{2} l^6(z). \quad (4.25)$$

The stringy $k = 2$ -instanton partition function in (4.4) receives these non-perturbative α' corrections, whereas the equivariant three-point functions $\langle D, Y_2^{(1)}, Y_2^{(1)} \rangle$, $\langle D, Y_2^{(2)}, Y_2^{(2)} \rangle$ computed in (C.11), (C.12) are trivial for $k = 2$, and thus we argue that the non-perturbative α' corrections in each coefficient are determined by the equivariant three-point functions as

$$K_{1;3}^{(5)}(z) : \langle \epsilon_1 \epsilon_2 D, D, D \rangle, \quad K_{1;3}^{(6)}(z) : \langle \epsilon_1 \epsilon_2 \epsilon D, D, D \rangle. \quad (4.26)$$

We also find that the classical intersection numbers obtained by the above computation completely coincide with the equivariant classical intersection numbers (C.8) of the divisor class D in $H_T^*(\text{Hilb}_k(\mathbb{C}^2), \mathbb{Q})$.

4.2 Stringy U(2) instanton counting

In the $N = 2$ case, let us consider a normalized partition function on $\mathcal{M}_{k,2}$ [27]:

$$\widehat{Z}_{k,2}^{\text{norm}}(\epsilon_1, \epsilon_2, \vec{a}, z) = (z\bar{z})^{i\frac{k}{2}r\epsilon} Z_{k,2}^{\text{norm}}(\epsilon_1, \epsilon_2, \vec{a}, z). \quad (4.27)$$

This normalization shifts a_α to $a_\alpha + \epsilon/2$. In the following, by putting $\zeta(s) = 0$ we ignore the perturbative α' corrections which depend on the second normalization factor in (4.2). Then in the asymptotic expansion around $r = 0$, we only need to consider the leading term of Z_L in (2.6). Up to $k = 4$ (instanton), we find that the asymptotic expansion has the form

$$\begin{aligned} \widehat{Z}_{k,2}^{\text{norm}}(\epsilon_1, \epsilon_2, \vec{a}, z) &= \sum_{\ell=0}^3 (-1)^\ell G_{k,\ell}^{(2)}(\epsilon_1, \epsilon_2, \vec{a}) l^{2\ell}(z) \\ &\quad - \frac{\epsilon}{\epsilon_1 \epsilon_2} G_{k-1,0}^{(2)}(\epsilon_1, \epsilon_2, \vec{a}) \left[e^{-K_2^{(3)}(z)} + \epsilon e^{-K_2^{(4)}(z)} + \epsilon^2 e^{-K_2^{(5)}(z)} + \epsilon^3 e^{-K_2^{(6)}(z)} \right] \\ &\quad + \frac{2\epsilon}{\epsilon_1^2 \epsilon_2^2} G_{k-2,0}^{(2)}(\epsilon_1, \epsilon_2, \vec{a}) \left[e^{-K_{2;2}^{(5)}(z)} + \epsilon e^{-K_{2;2}^{(6)}(z)} \right] \\ &\quad + 2\epsilon G_{k,2}^{(2)}(\epsilon_1, \epsilon_2, \vec{a}) \left[e^{-K_{2;3}^{(5)}(z)} + \epsilon e^{-K_{2;3}^{(6)}(z)} \right] + \mathcal{O}(r^{-4k+7}), \end{aligned} \quad (4.28)$$

where $\epsilon_{1,2} = ir\epsilon_{1,2}$, $\epsilon = \epsilon_1 + \epsilon_2$, $\vec{a} = ir\vec{a}$, and $l^n(z) = \frac{1}{n!} \log^n z\bar{z}$. We see that by the normalization factor $(z\bar{z})^{i\frac{k}{2}r\epsilon}$ in (4.27), the odd power terms with $l^{2\ell+1}$ for $\epsilon = 0$ have been removed from the first line of (4.28). Here $G_{k,\ell}^{(2)}$ behaves as $\mathcal{O}(r^{-4k+2\ell})$, and its leading term is given by the Nekrasov partition function (2.12):

$$G_{k,0}^{(2)}(\epsilon_1, \epsilon_2, \vec{a}) = Z_{k,2}^{\text{Nek}}(\epsilon_1, \epsilon_2, \vec{a}), \quad Z_{k<0,2}^{\text{Nek}}(\epsilon_1, \epsilon_2, \vec{a}) \equiv 0. \quad (4.29)$$

Up to $k = 4$, one has $Z_{0,2}^{\text{Nek}}(\epsilon_1, \epsilon_2, \vec{a}) = 1$,

$$Z_{1,2}^{\text{Nek}}(\epsilon_1, \epsilon_2, \vec{a}) = \frac{2}{\epsilon_1 \epsilon_2 D_{1,1}}, \quad (4.30)$$

$$Z_{2,2}^{\text{Nek}}(\epsilon_1, \epsilon_2, \vec{a}) = \frac{8\epsilon_1^2 + 17\epsilon_1 \epsilon_2 + 8\epsilon_2^2 - 2\mathbf{a}^2}{\epsilon_1^2 \epsilon_2^2 D_{1,1} D_{1,2} D_{2,1}}, \quad (4.31)$$

$$Z_{3,2}^{\text{Nek}}(\epsilon_1, \epsilon_2, \vec{a}) = \frac{N_3}{3\epsilon_1^3 \epsilon_2^3 D_{1,1} D_{1,2} D_{2,1} D_{1,3} D_{3,1}}, \quad (4.32)$$

$$Z_{4,2}^{\text{Nek}}(\epsilon_1, \epsilon_2, \vec{a}) = \frac{N_4}{6\epsilon_1^4 \epsilon_2^4 D_{1,1} D_{1,2} D_{2,1} D_{1,3} D_{3,1} D_{2,2} D_{1,4} D_{4,1}}. \quad (4.33)$$

Here $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2$, and $D_{i,j} = D_{i,j}(\epsilon_1, \epsilon_2, \vec{a})$, $N_3 = N_3(\epsilon_1, \epsilon_2, \vec{a})$, $N_4 = N_4(\epsilon_1, \epsilon_2, \vec{a})$ are given by

$$D_{i,j} = (i\epsilon_1 + j\epsilon_2)^2 - \mathbf{a}^2, \quad (4.34)$$

$$N_3 = 2(72(\epsilon_1^4 + \epsilon_2^4) + 363(\epsilon_1^3 \epsilon_2 + \epsilon_1 \epsilon_2^3) + 594\epsilon_1^2 \epsilon_2^2 - (26\epsilon_1^2 + 47\epsilon_1 \epsilon_2 + 26\epsilon_2^2)\mathbf{a}^2 + 2\mathbf{a}^4), \quad (4.35)$$

$$\begin{aligned}
 N_4 = & 9216(\varepsilon_1^8 + \varepsilon_2^8) + 100608(\varepsilon_1^7\varepsilon_2 + \varepsilon_1\varepsilon_2^7) + 440688(\varepsilon_1^6\varepsilon_2^2 + \varepsilon_1^2\varepsilon_2^6) + 1009131(\varepsilon_1^5\varepsilon_2^3 + \varepsilon_1^3\varepsilon_2^5) \\
 & + 1319994\varepsilon_1^4\varepsilon_2^4 - (6208(\varepsilon_1^6 + \varepsilon_2^6) + 44336(\varepsilon_1^5\varepsilon_2 + \varepsilon_1\varepsilon_2^5) + 124139(\varepsilon_1^4\varepsilon_2^2 + \varepsilon_1^2\varepsilon_2^4) \\
 & + 171845\varepsilon_1^3\varepsilon_2^3)\mathfrak{a}^2 + (1440(\varepsilon_1^4 + \varepsilon_2^4) + 5644(\varepsilon_1^3\varepsilon_2 + \varepsilon_1\varepsilon_2^3) + 8651\varepsilon_1^2\varepsilon_2^2)\mathfrak{a}^4 \\
 & - (132(\varepsilon_1^2 + \varepsilon_2^2) + 212\varepsilon_1\varepsilon_2)\mathfrak{a}^6 + 4\mathfrak{a}^8.
 \end{aligned} \tag{4.36}$$

We find that $G_{k,1}^{(2)}$ is also given by the Nekrasov partition function

$$G_{k,1}^{(2)}(\varepsilon_1, \varepsilon_2, \vec{\mathfrak{a}}) = \frac{1}{2\varepsilon_1\varepsilon_2} Z_{k-1,2}^{\text{Nek}}(\varepsilon_1, \varepsilon_2, \vec{\mathfrak{a}}). \tag{4.37}$$

Up to $k = 4$, $G_{k,2}^{(2)}$ are given by

$$G_{1,2}^{(2)}(\varepsilon_1, \varepsilon_2, \vec{\mathfrak{a}}) = \frac{\varepsilon^2 - \mathfrak{a}^2}{8\varepsilon_1\varepsilon_2}, \tag{4.38}$$

$$G_{2,2}^{(2)}(\varepsilon_1, \varepsilon_2, \vec{\mathfrak{a}}) = \frac{\varepsilon_1^2 + 3\varepsilon_1\varepsilon_2 + \varepsilon_2^2 - \mathfrak{a}^2}{\varepsilon_1^2\varepsilon_2^2 D_{1,1}}, \tag{4.39}$$

$$G_{3,2}^{(2)}(\varepsilon_1, \varepsilon_2, \vec{\mathfrak{a}}) = \frac{N_{3,2}}{8\varepsilon_1^3\varepsilon_2^3 D_{1,1} D_{1,2} D_{2,1}}, \tag{4.40}$$

$$G_{4,2}^{(2)}(\varepsilon_1, \varepsilon_2, \vec{\mathfrak{a}}) = \frac{N_{4,2}}{3\varepsilon_1^4\varepsilon_2^4 D_{1,1} D_{1,2} D_{2,1} D_{1,3} D_{3,1}}, \tag{4.41}$$

where

$$N_{3,2} = 56(\varepsilon_1^4 + \varepsilon_2^4) + 337(\varepsilon_1^3\varepsilon_2 + \varepsilon_1\varepsilon_2^3) + 582\varepsilon_1^2\varepsilon_2^2 - (70\varepsilon_1^2 + 133\varepsilon_1\varepsilon_2 + 70\varepsilon_2^2)\mathfrak{a}^2 + 14\mathfrak{a}^4, \tag{4.42}$$

$$\begin{aligned}
 N_{4,2} = & 180(\varepsilon_1^6 + \varepsilon_2^6) + 1767(\varepsilon_1^5\varepsilon_2 + \varepsilon_1\varepsilon_2^5) + 6018(\varepsilon_1^4\varepsilon_2^2 + \varepsilon_1^2\varepsilon_2^4) + 8936\varepsilon_1^3\varepsilon_2^3 - (245(\varepsilon_1^4 + \varepsilon_2^4) \\
 & + 1062(\varepsilon_1^3\varepsilon_2 + \varepsilon_1\varepsilon_2^3) + 1676\varepsilon_1^2\varepsilon_2^2)\mathfrak{a}^2 + (70(\varepsilon_1^2 + \varepsilon_2^2) + 111\varepsilon_1\varepsilon_2)\mathfrak{a}^4 - 5\mathfrak{a}^6.
 \end{aligned} \tag{4.43}$$

Up to $k = 4$, $G_{k,3}^{(2)}$ are given by

$$G_{1,3}^{(2)}(\varepsilon_1, \varepsilon_2, \vec{\mathfrak{a}}) = \frac{(\varepsilon^2 - \mathfrak{a}^2)^2}{32\varepsilon_1\varepsilon_2}, \tag{4.44}$$

$$G_{2,3}^{(2)}(\varepsilon_1, \varepsilon_2, \vec{\mathfrak{a}}) = \frac{N_{2,3}}{\varepsilon_1^2\varepsilon_2^2 D_{1,1}}, \tag{4.45}$$

$$G_{3,3}^{(2)}(\varepsilon_1, \varepsilon_2, \vec{\mathfrak{a}}) = \frac{N_{3,3}}{32\varepsilon_1^3\varepsilon_2^3 D_{1,1} D_{1,2} D_{2,1}}, \tag{4.46}$$

$$G_{4,3}^{(2)}(\varepsilon_1, \varepsilon_2, \vec{\mathfrak{a}}) = \frac{N_{4,3}}{3\varepsilon_1^4\varepsilon_2^4 D_{1,1} D_{1,2} D_{2,1} D_{1,3} D_{3,1}}, \tag{4.47}$$

where

$$N_{2,3} = (\varepsilon_1^4 + \varepsilon_2^4) + 10(\varepsilon_1^3\varepsilon_2 + \varepsilon_1\varepsilon_2^3) + 19\varepsilon_1^2\varepsilon_2^2 - 2(\varepsilon_1^2 + 5\varepsilon_1\varepsilon_2 + \varepsilon_2^2)\mathfrak{a}^2 + \mathfrak{a}^4, \tag{4.48}$$

$$\begin{aligned}
 N_{3,3} = & 488(\varepsilon_1^6 + \varepsilon_2^6) + 5905(\varepsilon_1^5\varepsilon_2 + \varepsilon_1\varepsilon_2^5) + 22244(\varepsilon_1^4\varepsilon_2^2 + \varepsilon_1^2\varepsilon_2^4) + 34310\varepsilon_1^3\varepsilon_2^3 \\
 & - (1098(\varepsilon_1^4 + \varepsilon_2^4) + 7754(\varepsilon_1^3\varepsilon_2 + \varepsilon_1\varepsilon_2^3) + 12392\varepsilon_1^2\varepsilon_2^2)\mathfrak{a}^2 \\
 & + (732(\varepsilon_1^2 + \varepsilon_2^2) + 1849\varepsilon_1\varepsilon_2)\mathfrak{a}^4 - 122\mathfrak{a}^6,
 \end{aligned} \tag{4.49}$$

$$\begin{aligned}
 N_{4,3} = & 612(\varepsilon_1^8 + \varepsilon_2^8) + 9867(\varepsilon_1^7 \varepsilon_2 + \varepsilon_1 \varepsilon_2^7) + 57891(\varepsilon_1^6 \varepsilon_2^2 + \varepsilon_1^2 \varepsilon_2^6) + 160209(\varepsilon_1^5 \varepsilon_2^3 + \varepsilon_1^3 \varepsilon_2^5) \\
 & + 224058 \varepsilon_1^4 \varepsilon_2^4 - (1445(\varepsilon_1^6 + \varepsilon_2^6) + 14434(\varepsilon_1^5 \varepsilon_2 + \varepsilon_1 \varepsilon_2^5) + 45445(\varepsilon_1^4 \varepsilon_2^2 + \varepsilon_1^2 \varepsilon_2^4) \\
 & + 65416 \varepsilon_1^3 \varepsilon_2^3) \mathfrak{a}^2 + (1071(\varepsilon_1^4 + \varepsilon_2^4) + 5003(\varepsilon_1^3 \varepsilon_2 + \varepsilon_1 \varepsilon_2^3) + 7729 \varepsilon_1^2 \varepsilon_2^2) \mathfrak{a}^4 \\
 & - (255(\varepsilon_1^2 + \varepsilon_2^2) + 436 \varepsilon_1 \varepsilon_2) \mathfrak{a}^6 + 17 \mathfrak{a}^8.
 \end{aligned} \tag{4.50}$$

Up to the precoefficients of $\zeta(s)$ given by the ratio of two homogeneous polynomials of $\varepsilon_{1,2}$ and \mathfrak{a} which depend on the regularization scheme, we find that the coefficients in (4.28) coincide with the coefficients in (4.4) for the stringy $U(1)$ instanton partition function:

$$e^{-K_2^{(n)}(z)} \equiv e^{-K_1^{(n)}(z)}, \quad e^{-K_{2,2}^{(d)}(z)} \equiv e^{-K_{1,2}^{(d)}(z)}, \quad e^{-K_{2,3}^{(d)}(z)} \equiv e^{-K_{1,3}^{(d)}(z)}, \tag{4.51}$$

where $n = 3, \dots, 6$ and $d = 5, 6$.

By combining the above results with the results in section 4.1, we can conjecture the quantum corrections of equivariant three-point functions on $\mathcal{M}_{k,2}$ up to the normalization constants determined by the classical intersection numbers. Let D be the Poincaré dual observable of the divisor class in the T -equivariant cohomology $H_T^*(\mathcal{M}_{k,2})$ of $\mathcal{M}_{k,2}$, where T is the $2 + 2$ dimensional torus as in footnote 6 of section 2.2. From the precoefficient of $e^{-K_2^{(3)}(z)}$ in (4.28) which does not depend on the normalization (4.27), by (4.14) we find

$$\langle D, D, D \rangle = -\frac{c_1 \epsilon}{\epsilon_1 \epsilon_2} Z_{k-1,2}^{\text{Nek}}(\varepsilon_1, \varepsilon_2, \vec{\mathfrak{a}}) \left(\frac{1}{2} + L_0(z) \right), \tag{4.52}$$

where c_1 is a constant, and $L_0(z) = z/(1-z)$. From the precoefficient of $e^{-K_{2,2}^{(5)}(z)}$ in (4.28), by (4.19) we also find that there exists a cohomological observable $Y_2^{(2)}$ dual to a (complex) codimension two cycle class in $H_T^*(\mathcal{M}_{k,2})$ with the quantum corrections in three-point function as

$$\langle D, Y_2^{(2)}, Y_2^{(2)} \rangle = \frac{c_2 \epsilon}{2 \epsilon_1^2 \epsilon_2^2} Z_{k-2,2}^{\text{Nek}}(\varepsilon_1, \varepsilon_2, \vec{\mathfrak{a}}) \left(\frac{1}{2} + L_0(z) \right), \tag{4.53}$$

where c_2 is a constant. Up to $k = 4$, there are no other non-trivial Poincaré dual observables Y of codimension two cycle classes with the quantum corrections in the three-point function $\langle D, Y, Y \rangle$.

In appendix D, we will describe the quantum cohomology ring $H_T^*(\mathcal{M}_{k,N})$ by the tensor product of Fock spaces [47, 52], and by computing the quantum parts $\langle D, Y, Y \rangle^{\text{qu}}$ of three-point functions, we see that the above results (4.52) and (4.53) are surely consistent with (D.12) and (D.13), respectively.

4.3 Stringy $U(3)$ instanton counting

In the $N = 3$ case, as in the $N = 2$ case we consider a normalization

$$\widehat{Z}_{k,3}^{\text{norm}}(\epsilon_1, \epsilon_2, \vec{\mathfrak{a}}, z) = (z\bar{z})^{i\frac{k}{2}r\epsilon} Z_{k,3}^{\text{norm}}(\epsilon_1, \epsilon_2, \vec{\mathfrak{a}}, z), \tag{4.54}$$

and ignore the perturbative α' corrections by putting $\zeta(s) = 0$. As in section 4.2, this normalization shifts a_α to $a_\alpha + \epsilon/2$. Up to $k = 3$ (-instanton), we find the asymptotic

expansion around $r = 0$ as

$$\begin{aligned}\widehat{Z}_{k,3}^{\text{norm}}(\epsilon_1, \epsilon_2, \vec{a}, z) &= \sum_{\ell=0}^3 (-1)^\ell G_{k,\ell}^{(3)}(\epsilon_1, \epsilon_2, \vec{a}) l^{2\ell}(z) - \sum_{\ell=0}^2 (-1)^\ell H_{k,\ell}^{(3)}(\epsilon_1, \epsilon_2, \vec{a}) l^{2\ell+1}(z) \\ &\quad + \frac{\varepsilon}{\varepsilon_1 \varepsilon_2} G_{k-1,0}^{(3)}(\epsilon_1, \epsilon_2, \vec{a}) \left[e^{-K_{3;2}^{(5)}(z)} + \varepsilon e^{-K_{3;2}^{(6)}(z)} \right] \\ &\quad - \frac{3\varepsilon}{\varepsilon_1 \varepsilon_2} H_{k-1,0}^{(3)}(\epsilon_1, \epsilon_2, \vec{a}) l^1(z) e^{-K_{3;2}^{(5)}(z)} + \mathcal{O}(r^{-6k+7}),\end{aligned}\quad (4.55)$$

where $\varepsilon_{1,2} = ir\epsilon_{1,2}$, $\varepsilon = \varepsilon_1 + \varepsilon_2$, $\vec{a} = ir\vec{a}$, and $l^n(z) = \frac{1}{n!} \log^n z \bar{z}$. Here $G_{k,\ell}^{(3)}$ behaves as $\mathcal{O}(r^{-6k+2\ell})$, and the leading term $G_{k,0}^{(3)}$ coincides with the Nekrasov partition function (2.12):

$$G_{k,0}^{(3)}(\varepsilon_1, \varepsilon_2, \vec{a}) = Z_{k,3}^{\text{Nek}}(\varepsilon_1, \varepsilon_2, \vec{a}), \quad Z_{k<0,3}^{\text{Nek}}(\varepsilon_1, \varepsilon_2, \vec{a}) \equiv 0. \quad (4.56)$$

For example, $Z_{0,3}^{\text{Nek}}(\varepsilon_1, \varepsilon_2, \vec{a}) = 1$ and

$$Z_{1,3}^{\text{Nek}}(\varepsilon_1, \varepsilon_2, \vec{a}) = \frac{2(3\varepsilon^2 + \mathbf{a}_{12}\mathbf{a}_{23} + \mathbf{a}_{23}\mathbf{a}_{31} + \mathbf{a}_{31}\mathbf{a}_{12})}{\varepsilon_1 \varepsilon_2 (\varepsilon^2 - \mathbf{a}_{12}^2)(\varepsilon^2 - \mathbf{a}_{23}^2)(\varepsilon^2 - \mathbf{a}_{31}^2)}, \quad (4.57)$$

where $\mathbf{a}_{\alpha\beta} = \mathbf{a}_\alpha - \mathbf{a}_\beta$. $H_{k,\ell}^{(3)}$ ($H_{k\leq 0,\ell}^{(3)} \equiv 0$) behaves as $\mathcal{O}(r^{-6k+2\ell+1})$, and for example

$$H_{1,0}^{(3)} = \frac{(\mathbf{a}_{12} - \mathbf{a}_{23})(\mathbf{a}_{23} - \mathbf{a}_{31})(\mathbf{a}_{31} - \mathbf{a}_{12})}{3\varepsilon_1 \varepsilon_2 (\varepsilon^2 - \mathbf{a}_{12}^2)(\varepsilon^2 - \mathbf{a}_{23}^2)(\varepsilon^2 - \mathbf{a}_{31}^2)}. \quad (4.58)$$

Up to the perturbative α' corrections we find $e^{-K_{3;2}^{(5)}(z)} \equiv e^{-K_{1;2}^{(5)}(z)} \equiv e^{-K_{2;2}^{(5)}(z)}$, and the coefficient $e^{-K_{3;2}^{(6)}(z)}$ is given by

$$\begin{aligned}e^{-K_{3;2}^{(6)}(z)} &\equiv \frac{3}{4} l^6(z) + \frac{3}{2} (L_3(z) + \text{c.c.}) l^3(z) - \frac{3}{2} \left(2L_4(z) - \int \int \frac{dz^2}{z^2} L_1(z)^2 + \text{c.c.} \right) l^2(z) \\ &\quad + \frac{3}{2} \left(2L_5(z) - \int \int \int \frac{dz^3}{z^3} L_1(z)^2 - \int \frac{dz}{z} L_2(z)^2 + \text{c.c.} \right) l^1(z) \\ &\quad + \frac{3}{2} (L_3(z) + \text{c.c.})^2.\end{aligned}\quad (4.59)$$

As in the $N = 2$ case, we find that there exist the observable D for the divisor class and an observable $Y_2^{(2)}$ for a (complex) codimension two dual cycle class in $H_T^*(\mathcal{M}_{k,3})$ with the quantum corrections in three-point function as

$$\langle D, Y_2^{(2)}, Y_2^{(2)} \rangle = \frac{c\varepsilon}{\varepsilon_1 \varepsilon_2} Z_{k-1,3}^{\text{Nek}}(\varepsilon_1, \varepsilon_2, \vec{a}) \left(\frac{1}{2} + L_0(z) \right), \quad (4.60)$$

where c is a constant. Here $H_T^*(\mathcal{M}_{k,3})$ is the T -equivariant cohomology of $\mathcal{M}_{k,3}$ with the $3+2$ dimensional torus action T mentioned in footnote 6. By the expansion (4.55), we see that the three-point function $\langle D, D, D \rangle$ does not have quantum corrections.

As same as the $N = 2$ case, we see that the above result (4.60) is consistent with the result (D.12) in appendix D.

5 Conclusion and discussions

In this paper, we have studied the stringy instanton partition function of four dimensional $\mathcal{N} = 2$ $U(N)$ supersymmetric gauge theory given in [27]. In section 3.1, we found that the stringy instanton partition function whose α' corrections have been removed coincides with the four dimensional limit (3.8) of the K-theoretic instanton partition function with a five dimensional Chern-Simons coefficient. Here the Kähler modulus ζ of the ADHM moduli space was obtained from the strong coupling limit of the five dimensional Chern-Simons term. We also discussed that the classical stringy instanton partition function is embedded to (refined) topological string theory on the local toric Calabi-Yau threefolds labeled by $m \in \mathbb{Z}$ in section 3.2. This gives geometric engineering of the instantons with classical stringy corrections, and we provided a physical explanation of this realization in section 3.3

We further studied the stringy instanton partition function with the α' corrections for $U(1)$, $U(2)$, and $U(3)$ cases. We found that the quantum stringy corrections have the universal structure for arbitrary instanton charge k , as in (4.4), (4.28), and (4.55). This universal structure shows that the stringy corrections for an instanton charge k contain the stringy corrections for instanton charges $l \leq k$. From the viewpoint of the Fock space description of the equivariant cohomology ring of the ADHM moduli space $\mathcal{M}_{k,N}$ in appendix C and D, such structure is clear from the construction of the elements of the equivariant cohomology by the generators of the Heisenberg algebra. Note that this structure seems to be related with a compactification of the UV non-compactness by point-like instantons, which is called the Uhlenbeck compactification of the framed moduli space $\mathcal{M}_{k,N}^{(0)}$ of point-like instantons on $S^4 \cong \mathbb{C}^2 \cup \{\infty\}$ (see e.g. [6, 10, 64]):¹⁰

$$\widehat{\mathcal{M}}_{k,N} = \bigsqcup_{l=0}^k \mathcal{M}_{l,N}^{(0)} \times S^{k-l}\mathbb{C}^2,$$

where $S^l\mathbb{C}^2$ is the l -th symmetric product of \mathbb{C}^2 . After resolving the orbifold singularities in $\widehat{\mathcal{M}}_{k,N}$, one obtains $\mathcal{M}_{k,N}$.

For $U(1)$ case, as discussed in [27] we read off some equivariant three-point functions on $\mathcal{M}_{k,1} \cong \text{Hilb}_k(\mathbb{C}^2)$, and confirmed the agreement with the computations in appendix C. Using this result, we extracted some equivariant three-point functions on $\mathcal{M}_{k,2}$ and $\mathcal{M}_{k,3}$ from the stringy instanton partition functions, and also checked that these results are consistent with the computations in appendix D.

It would be interesting to further study the classical and the quantum structure of the $U(N)$ stringy instanton partition function, and to compare the structure with the discussion of the quantum multiplication for the ADHM moduli space $\mathcal{M}_{k,N}$ (as described in appendix D) [47]. The refined topological vertex [21, 22] do not capture the quantum stringy corrections, and thus it would be also interesting to formulate “quantum refined topological vertex” which captures such quantum corrections.

¹⁰We would like to thank the referee for pointing out this similarity.

A six dimensional analogue of the four dimensional instanton partition function was discussed in [72–74]. It would be interesting to discuss the stringy generalization of the six dimensional instanton partition function.

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A Multiple gamma function

The multiple gamma function is defined by

$$\Gamma_r(x|\omega_1, \dots, \omega_r) = \exp \left(\left. \frac{\partial}{\partial s} \right|_{s=0} \zeta_r(s, x|\omega_1, \dots, \omega_r) \right), \quad (\text{A.1})$$

where ζ_r is the Barnes zeta function defined by the analytic continuation of an infinite sum

$$\zeta_r(s, x|\omega_1, \dots, \omega_r) = \sum_{n_1, \dots, n_r=0}^{\infty} \frac{1}{(x + n_1\omega_1 + \dots + n_r\omega_r)^s}, \quad \text{Re}(s) > r. \quad (\text{A.2})$$

A.1 Zeta function regularization

Using the multiple gamma function, one can regularize an infinite product by

$$\left[\prod_{n_1, \dots, n_r=0}^{\infty} (x + n_1\omega_1 + \dots + n_r\omega_r)^{-1} \right]_{\text{reg}} = \Gamma_r(x|\omega_1, \dots, \omega_r). \quad (\text{A.3})$$

The usual gamma function

$$\Gamma(x) = \int_0^{\infty} \frac{dt}{t^{1-x}} e^{-t}, \quad \text{Re}(x) > 0, \quad (\text{A.4})$$

is related with the modified gamma function Γ_1 by

$$\Gamma(x) = \sqrt{2\pi} \omega^{\frac{1}{2}-x} \Gamma_1(\omega x|\omega). \quad (\text{A.5})$$

Then the zeta function regularization used in [25, 26] is obtained

$$\left[\prod_{n=0}^{\infty} (x+n) \right]_{\text{reg}} = \frac{\sqrt{2\pi}}{\Gamma(x)}. \quad (\text{A.6})$$

A.2 Perturbative partition function of four dimensional gauge theory

Using a formal expansion

$$\frac{1}{1-e^x} = \sum_{n=0}^{\infty} e^{nx}, \quad (\text{A.7})$$

the perturbative partition function (3.3) is written as [11, 64]

$$\begin{aligned} Z_N^{\text{cD5}}(\epsilon_1, \epsilon_2, \vec{a}) &= \prod_{\substack{\alpha, \beta=1 \\ (\alpha \neq \beta)}}^N \Gamma_2(a_{\alpha\beta} | \epsilon_1, \epsilon_2) \\ &= \prod_{\substack{\alpha, \beta=1 \\ (\alpha \neq \beta)}}^N \exp \left(\left. \frac{\partial}{\partial s} \right|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1-s}} \frac{e^{-a_{\alpha\beta} t}}{(e^{-\epsilon_1 t} - 1)(e^{-\epsilon_2 t} - 1)} \right). \end{aligned} \quad (\text{A.8})$$

B Stringy U(1) instanton counting, simple Hurwitz theory, and topological strings

In this appendix, we discuss relations between the stringy U(1) instanton counting, the simple Hurwitz theory, and the topological A-model on local toric curve. Let us consider the generating function of the stringy U(1) k -instanton partition functions (2.5) for $N = 1$ on the (r -scaled) anti-self-dual Omega background $\epsilon_1 = -\epsilon_2 = \hbar/r$:¹¹

$$Z_{\text{U}(1)}^{\text{SI}}(\hbar, z, \mathbf{q}) \equiv 1 + \sum_{k=1}^{\infty} \left((z\bar{z})^{-ira} \mathbf{q}^2 \right)^k Z_{k,1}(\hbar/r, -\hbar/r, a, z) = \sum_{\mu} \left(\frac{\mathbf{q}}{\hbar} \right)^{2|\mu|} \frac{(z\bar{z})^{i\hbar \sum_{s \in \mu} c_{\mu}(s)}}{\prod_{s \in \mu} h_{\mu}(s)^2}, \quad (\text{B.1})$$

where $c_{\mu}(s) = \sum_{(i,j) \in \mu} (j-i)$ and $h_{\mu}(s) = a_{\mu}(s) + \ell_{\mu}(s) + 1$ is the hook length. By expanding this partition function around $\hbar = 0$:

$$Z_{\text{U}(1)}^{\text{SI}}(\hbar, z, \mathbf{q}) = \sum_{k, \ell=0}^{\infty} (i\hbar \log z\bar{z})^{2\ell-2k} \frac{(i\mathbf{q} \log z\bar{z})^{2k}}{(2\ell)!} G_{k, \ell}, \quad (\text{B.2})$$

one finds that the coefficients

$$G_{k, \ell} = \sum_{|\mu|=k} \frac{1}{\prod_{s \in \mu} h_{\mu}(s)^2} \left(\sum_{s \in \mu} c_{\mu}(s) \right)^{2\ell} \quad (\text{B.3})$$

coincide with the equivariant classical intersection numbers $\langle D^{2\ell} \rangle^{\text{cl}}|_{\epsilon=0}$ of the divisor class on the Hilbert scheme of points $\text{Hilb}_k(\mathbb{C}^2)$ on \mathbb{C}^2 as computed in (C.8). One also finds that

$$H_{g, k}^{\mathbb{P}^1} = G_{k, g-1+k} \quad (\text{B.4})$$

gives the disconnected simple Hurwitz number of \mathbb{P}^1 which counts the degree k ramified cover $f : \Sigma_g \rightarrow \mathbb{P}^1$ with $m = 2g - 2 + 2k$ simple branch points, where Σ_g is a genus g

¹¹In this background, as shown in (2.18), the stringy instanton partition function does not have the α' corrections, and thus (2.17) for $N = 1$ is obtained as the exact result.

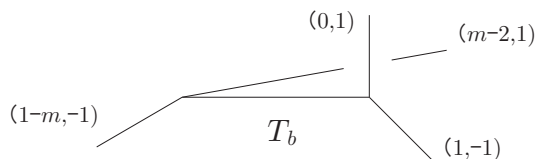


Figure 3. The toric diagram of the local curve $X_{1,m} = \mathcal{O}(m-2) \oplus \mathcal{O}(-m) \rightarrow \mathbb{P}^1$ (total space of rank two vector bundle over \mathbb{P}^1), where T_b is the Kähler modulus of \mathbb{P}^1 .

Riemann surface [75] (see also e.g. [76]).¹² Then the connected simple Hurwitz numbers $H_{g,k}^{\mathbb{P}^1 \bullet}$ are obtained from the genus expansion of the free energy

$$\mathcal{F}_{\text{U}(1)}^{\text{SI}}(\hat{h}, x) = \log Z_{\text{U}(1)}^{\text{SI}}(\hat{h}, z, \mathbf{q}) = \sum_{g=0}^{\infty} \hat{h}^{2g-2} \mathcal{F}_g(x) = \sum_{g=0}^{\infty} \hat{h}^{2g-2} \sum_{k=1}^{\infty} \frac{x^k}{(2g-2+2k)!} H_{g,k}^{\mathbb{P}^1 \bullet}, \quad (\text{B.5})$$

where $\hat{h} = i\hbar \log z\bar{z}$ and $x = (i\mathbf{q} \log z\bar{z})^2$. We see that the perturbative free energies $\mathcal{F}_g(x)$ are given as [77, 78]

$$\mathcal{F}_0(x) = \sum_{k=1}^{\infty} \frac{k^{k-3}}{k!} x^k, \quad (\text{B.6})$$

$$\mathcal{F}_1(x) = -\frac{1}{24} (\log(1-y) + y), \quad (\text{B.7})$$

$$\mathcal{F}_{g \geq 2}(x) = \frac{y^2}{(1-y)^{5(g-1)}} \sum_{i=1}^{3g-5} c_{g,i} y^i, \quad (\text{B.8})$$

where $c_{g,i}$ are constants,¹³ and $y = -W(-x)$ is the Lambert W function defined by the inverse function of the spectral curve $x = ye^{-y}$ which has the series expansion $y = \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} x^k$. The relation between the Hurwitz numbers of \mathbb{P}^1 and the intersection numbers on $\text{Hilb}_k(\mathbb{C}^2)$ was discussed in [79].

As mentioned in section 3.2, the stringy $\text{U}(1)$ instanton partition function (B.1) is obtained from the A-model partition function $Z_{X_{1,m}}^{\text{topA}}$ on the local curve $X_{1,m} = \mathcal{O}(m-2) \oplus \mathcal{O}(-m) \rightarrow \mathbb{P}^1$ described in figure 3. By the geometric engineering $e^{-T_b} \sim (\beta\mathbf{q})^2$, $g_s \sim \hbar$, and the four dimensional limit (3.8): $\beta \rightarrow 0$, $m \rightarrow \infty$ with fixed $\beta m = r \log z\bar{z}$, we have

$$Z_{X_{1,m}}^{\text{topA}}(g_s, T_b) \longrightarrow Z_{\text{U}(1)}^{\text{SI}}(\hat{h}, z, \mathbf{q}). \quad (\text{B.9})$$

¹²The simple branch point is a branch point such that the branching number is one, and the number m of simple branch points is determined by the Riemann-Hurwitz formula $2g-2+2k = \sum_{i=1}^m (k - \ell_{\mu^i})$, where $\mu^i = (\mu_1^i \geq \mu_2^i \geq \dots \geq \mu_{\ell_{\mu^i}}^i > 0)$ is a profile over an i -th branch point. Here the profile of the simple branch point is given by $\mu = (2, 1^{k-2})$. Note that the genus g of the disconnected simple Hurwitz numbers can be a negative integer.

¹³For example these are given by

$$\mathcal{F}_2(x) = \frac{y^2}{1440(1-y)^5} (6y+1), \quad \mathcal{F}_3(x) = \frac{y^2}{725760(1-y)^{10}} (720y^4 + 3816y^3 + 3482y^2 + 548y + 9).$$

Therefore one also obtains the relation between the topological A-model on $X_{1,m}$ and the simple Hurwitz theory which was previously discussed in [48]. As pointed out in [48], the simple Hurwitz theory is related with the $U(1)$ instanton counting with the first and second Casimir operators (see also [80]).¹⁴ Then we see that the stringy $U(1)$ instanton partition function (B.1) also coincides with the $U(1)$ instanton partition function with the first and second Casimir operators.

C Equivariant correlators on $\text{Hilb}_k(\mathbb{C}^2)$

Let $\text{Hilb}_k(\mathbb{C}^2)$ be the Hilbert scheme of points on \mathbb{C}^2 :

$$\text{Hilb}_k(\mathbb{C}^2) = \{\mathcal{J} \subset \mathbb{C}[x, y] | \mathcal{J} \text{ is an ideal, } \dim_{\mathbb{C}} \mathbb{C}[x, y]/\mathcal{J} = k\}. \quad (\text{C.1})$$

The T -equivariant cohomology $H_T^*(\text{Hilb}_k(\mathbb{C}^2), \mathbb{Q})$ of $\text{Hilb}_k(\mathbb{C}^2)$ with the equivariant action $T = U(1)^2$ on \mathbb{C}^2 has the Fock space description over \mathbb{Q} as was constructed by Grojnowski and Nakajima [44, 49, 50] (see also [51]). The Fock module over the Heisenberg algebra $\{\alpha_{\pm n}, n \in \mathbb{N} \mid [\alpha_m, \alpha_n] = m\delta_{m+n,0}\}$ is given as follows. The Fock vacuum $|\emptyset\rangle$ is annihilated by $\alpha_{n>0}$: $\alpha_n|\emptyset\rangle = 0$ for $n > 0$, and the basis of the Fock space \mathcal{F} is created by $\alpha_{n<0}$:

$$|Y\rangle = \frac{1}{|\text{Aut}(Y)| \prod_i Y_i} \prod_i \alpha_{-Y_i} |\emptyset\rangle. \quad (\text{C.2})$$

Here Y is a partition with $Y_1 \geq Y_2 \geq \dots \geq Y_{\ell_Y} > 0$, and $\text{Aut}(Y)$ is the order of the automorphism group of the partition. Then a Fock module $|Y\rangle$ with $|Y| = \sum_{i=1}^{\ell_Y} Y_i = k$ gives an element of $H_T^{2k-2\ell_Y}(\text{Hilb}_k(\mathbb{C}^2), \mathbb{Q})$, and a canonical isomorphism

$$\mathcal{F} \otimes_{\mathbb{Q}} \mathbb{Q}[\epsilon_1, \epsilon_2] \cong \bigoplus_{k \geq 0} H_T^*(\text{Hilb}_k(\mathbb{C}^2), \mathbb{Q}) \quad (\text{C.3})$$

is obtained, where $\epsilon_{1,2}$ are the equivariant parameters. The inner product on the Fock space which gives the equivariant two-point functions on $\text{Hilb}_k(\mathbb{C}^2)$ is normalized as

$$\langle Y | Y' \rangle = \frac{(-1)^{|Y|-\ell_Y}}{(\epsilon_1 \epsilon_2)^{\ell_Y} |\text{Aut}(Y)| \prod_i Y_i} \delta_{YY'}. \quad (\text{C.4})$$

The Poincaré dual of the divisor class in $H_T^*(\text{Hilb}_k(\mathbb{C}^2), \mathbb{Q})$ is given by $|D\rangle = -|2, 1^{k-2}\rangle$. The operator of small quantum multiplication by D is given by the q -deformed Calogero-Sutherland Hamiltonian [51]

$$\begin{aligned} H_D = & \frac{\epsilon}{2} \sum_{n=2}^{\infty} \left(\frac{1-q}{1+q} - \frac{1+(-q)^n}{1-(-q)^n} n \right) \alpha_{-n} \alpha_n \\ & + \frac{1}{2} \sum_{m,n=1}^{\infty} (\epsilon_1 \epsilon_2 \alpha_{-m} \alpha_{-n} \alpha_{m+n} - \alpha_{-m-n} \alpha_m \alpha_n), \end{aligned} \quad (\text{C.5})$$

where $\epsilon = \epsilon_1 + \epsilon_2$.

¹⁴The (simple) Hurwitz theory is also related with the topological A-model on \mathbb{P}^1 [76, 79].

Using this Fock space description, let us compute the equivariant classical intersection numbers of the divisor class in $H_T^*(\text{Hilb}_k(\mathbb{C}^2), \mathbb{Q})$. Let $H_D^{\text{cl}} = H_D|_{q=0}$ be the classical part of the operator H_D . By $\alpha_n^\ell \alpha_{-n}^k |\emptyset\rangle = k(k-1)\cdots(k-\ell+1)n^\ell \alpha_{-n}^{k-\ell} |\emptyset\rangle$ for $\ell \leq k$, one obtains

$$\begin{aligned} H_D^{\text{cl}} \alpha_{-2} \alpha_{-1}^{k-2} |\emptyset\rangle = & -\epsilon \alpha_{-2} \alpha_{-1}^{k-2} |\emptyset\rangle + \epsilon_1 \epsilon_2 \alpha_{-1}^k |\emptyset\rangle - \frac{1}{2}(k-2)(k-3) \alpha_{-2}^2 \alpha_{-1}^{k-4} |\emptyset\rangle \\ & - 2(k-2) \alpha_{-3} \alpha_{-1}^{k-3} |\emptyset\rangle, \end{aligned} \quad (\text{C.6})$$

and

$$\begin{aligned} (H_D^{\text{cl}})^2 \alpha_{-2} \alpha_{-1}^{k-2} |\emptyset\rangle = & \left(\epsilon^2 - \frac{1}{2}(3k^2 + k - 12) \epsilon_1 \epsilon_2 \right) \alpha_{-2} \alpha_{-1}^{k-2} |\emptyset\rangle - \epsilon_1 \epsilon_2 \epsilon \alpha_{-1}^k |\emptyset\rangle \\ & + \frac{3}{2}(k-2)(k-3) \epsilon \alpha_{-2}^2 \alpha_{-1}^{k-4} |\emptyset\rangle + 8(k-2) \epsilon \alpha_{-3} \alpha_{-1}^{k-3} |\emptyset\rangle \\ & + \frac{1}{4}(k-2)(k-3)(k-4)(k-5) \alpha_{-2}^3 \alpha_{-1}^{k-6} |\emptyset\rangle \\ & + 3(k-2)(k-3)(k-4) \alpha_{-3} \alpha_{-2} \alpha_{-1}^{k-5} |\emptyset\rangle \\ & + 8(k-2)(k-3) \alpha_{-4} \alpha_{-1}^{k-4} |\emptyset\rangle. \end{aligned} \quad (\text{C.7})$$

Then on $\text{Hilb}_k(\mathbb{C}^2)$, the equivariant classical intersection numbers of the divisor class are computed as

$$\begin{aligned} \langle D^0 \rangle^{\text{cl}} = \langle D^0 \rangle &= \frac{1}{k! (\epsilon_1 \epsilon_2)^k}, & \langle D^1 \rangle^{\text{cl}} = \langle D^1 \rangle &= 0, \\ \langle D^2 \rangle^{\text{cl}} = \langle D^2 \rangle &= -\frac{1}{2(k-2)! (\epsilon_1 \epsilon_2)^{k-1}}, \\ \langle D^3 \rangle^{\text{cl}} = \langle D | H_D | D \rangle^{\text{cl}} &= \frac{\epsilon}{2(k-2)! (\epsilon_1 \epsilon_2)^{k-1}}, \\ \langle D^4 \rangle^{\text{cl}} = \langle D | H_D^2 | D \rangle^{\text{cl}} &= -\frac{\epsilon^2}{2(k-2)! (\epsilon_1 \epsilon_2)^{k-1}} + \frac{3k^2 + k - 12}{4(k-2)! (\epsilon_1 \epsilon_2)^{k-2}}, \\ \langle D^5 \rangle^{\text{cl}} = \langle D | H_D^3 | D \rangle^{\text{cl}} &= \frac{\epsilon^3}{2(k-2)! (\epsilon_1 \epsilon_2)^{k-1}} \\ &\quad - \frac{(18(k-2) + 2(k-2)(k-3) + (3k^2 + k - 12)) \epsilon}{2(k-2)! (\epsilon_1 \epsilon_2)^{k-2}}, \\ \langle D^6 \rangle^{\text{cl}} = \langle D | H_D^4 | D \rangle^{\text{cl}} &= -\frac{\epsilon^4}{2(k-2)! (\epsilon_1 \epsilon_2)^{k-1}} \\ &\quad + \frac{(180(k-2) + 16(k-2)(k-3) + 3(3k^2 + k - 12)) \epsilon^2}{4(k-2)! (\epsilon_1 \epsilon_2)^{k-2}} \\ &\quad - \frac{15k^4 + 30k^3 - 105k^2 - 700k + 1344}{8(k-2)! (\epsilon_1 \epsilon_2)^{k-3}}. \end{aligned} \quad (\text{C.8})$$

By $\epsilon_{1,2} \rightarrow -\epsilon_{1,2}$, these results coincide with the classical part of the stringy $U(1)$ instanton counting computed in (4.4). Especially from (4.6), one finds a formula of the classical intersection numbers at $\epsilon = \epsilon_1 + \epsilon_2 = 0$:

$$\langle D^{2\ell} \rangle^{\text{cl}} \Big|_{\epsilon=0} = \frac{1}{(\epsilon_1 \epsilon_2)^{k-\ell}} \sum_{|\mu|=k} \frac{1}{\prod_{s \in \mu} h_\mu(s)^2} \left(\sum_{s \in \mu} c_\mu(s) \right)^{2\ell}, \quad (\text{C.9})$$

where $c_\mu(s) = \sum_{(i,j) \in \mu} (j - i)$ and $h_\mu(s) = a_\mu(s) + \ell_\mu(s) + 1$ is the hook length defined for a Young diagram μ as described in figure 1 of section 2.1.

The (quantum) equivariant three-point functions in $H_T^*(\text{Hilb}_k(\mathbb{C}^2), \mathbb{Q})$ are also computed as [27]:

$$\begin{aligned} \langle D | H_D | D \rangle &= \epsilon \left(\frac{1-q}{1+q} - \frac{2(1+q^2)}{1-q^2} \right) \langle D | D \rangle \\ &= -\frac{\epsilon}{(k-2)!(\epsilon_1 \epsilon_2)^{k-1}} \left(\frac{1}{2} + L_0(q) \right), \end{aligned} \quad (\text{C.10})$$

$$\begin{aligned} \langle Y_2^{(1)} | H_D | Y_2^{(1)} \rangle &= -\frac{3}{2} \epsilon \left(\frac{1-q}{1+q} - \frac{3(1-q^3)}{1+q^2} \right) \langle Y_2^{(1)} | Y_2^{(1)} \rangle \\ &= \frac{\epsilon}{(k-3)!(\epsilon_1 \epsilon_2)^{k-2}} \left(1 + M_0(q) \right), \end{aligned} \quad (\text{C.11})$$

$$\begin{aligned} \langle Y_2^{(2)} | H_D | Y_2^{(2)} \rangle &= -2\epsilon \left(\frac{1-q}{1+q} - \frac{2(1+q^2)}{1-q^2} \right) \langle Y_2^{(2)} | Y_2^{(2)} \rangle \\ &= \frac{\epsilon}{2(k-4)!(\epsilon_1 \epsilon_2)^{k-2}} \left(\frac{1}{2} + L_0(q) \right), \end{aligned} \quad (\text{C.12})$$

where we have changed the equivariant parameters as $\epsilon_{1,2} \rightarrow -\epsilon_{1,2}$. Here $|Y_2^{(1)}\rangle = |3, 1^{k-3}\rangle$, $|Y_2^{(2)}\rangle = |2^2, 1^{k-4}\rangle$ are the cohomological classes dual to (complex) codimension two cycle classes, and

$$L_0(q) = \frac{q}{1-q}, \quad M_0(q) = \frac{q(1-2q)}{1-q+q^2}. \quad (\text{C.13})$$

D Equivariant correlators on $\mathcal{M}_{k,N}$

In this appendix, we discuss equivariant correlators on the ADHM moduli space $\mathcal{M}_{k,N}$ defined in (2.1). In [52], by extending the construction of Grojnowski and Nakajima [44, 49, 50] for $\mathcal{M}_{k,1} \cong \text{Hilb}_k(\mathbb{C}^2)$ mentioned in appendix C, Baranovsky constructed the action of the Heisenberg algebra on the cohomology of $\mathcal{M}_{k,N}$. The T -equivariant cohomology $\bigoplus_{k \geq 0} H_T^*(\mathcal{M}_{k,N})$, where T is the $N+2$ dimensional torus with the equivariant parameters $a_1, \dots, a_N, \epsilon_{1,2}$ as in footnote 6 of section 2.2, forms a tensor product [47, 52]:

$$\bigoplus_{k \geq 0} H_T^*(\mathcal{M}_{k,N}) = H_{T_i}^*(\mathcal{M}_1)^{\otimes N}, \quad H_{T_i}^*(\mathcal{M}_1) = \bigoplus_{k \geq 0} H_{T_i}^*(\mathcal{M}_{k,1}, \mathbb{Q}). \quad (\text{D.1})$$

Here T_i is the $1+2$ dimensional torus with the equivariant parameters a_i and $\epsilon_{1,2}$, where the torus e^{ia_i} trivially acts on the i -th $\mathcal{M}_{k,1}$. The Heisenberg algebra is given by $\{\alpha_{\pm n}^{(i)}, n \in \mathbb{N}, i = 1, \dots, N \mid [\alpha_m^{(i)}, \alpha_n^{(j)}] = m\delta_{m+n,0}\delta_{i,j}\}$, and the generators $\alpha_{\pm n}^{(i)}$ only act on the i -th tensor factor.

Let $|\mathbf{1}_k\rangle$ be the element in the tensor product of Fock spaces corresponding to the cohomological degree 0 element $\mathbf{1}_k$ in $H_T^0(\mathcal{M}_{k,N})$, and using the isomorphism (C.3), we consider

$$|Y\rangle = \frac{1}{|\text{Aut}(Y)|N \prod_I Y_I} \prod_I \beta_{-Y_I} |\mathbf{1}_{k-I}\rangle. \quad (\text{D.2})$$

Here Y is a partition with $Y_1 \geq Y_2 \geq \dots \geq Y_{\ell_Y} > 0$ and $|Y| = \sum_{I=1}^{\ell_Y} Y_I = l$, $\text{Aut}(Y)$ is the order of the automorphism group of the partition, and

$$\beta_{\pm n} = \sum_{i=1}^N \alpha_{\pm n}^{(i)} \quad (\text{D.3})$$

are the Baranovsky operators [52]. The Baranovsky operators act on the cohomology ring of $\mathcal{M}_{k,N}$ as

$$\beta_{\pm n} : \bigoplus_k H_T^*(\mathcal{M}_{k,N}) \longrightarrow \bigoplus_k H_T^*(\mathcal{M}_{k \mp n, N}), \quad (\text{D.4})$$

and the action $\beta_{-n}, n > 0$ increases the cohomological degree by

$$2nN - 2. \quad (\text{D.5})$$

Therefore the module (D.2) describes an element of $H_T^{2lN-2\ell_Y}(\mathcal{M}_{k,N})$. On the other hand, the action $\beta_n, n > 0$ decreases the cohomological degree by $2nN - 2$, and thus if $n \geq 2$, or $n = 1$ for $N \geq 2$, then $\beta_n 1_k = 0$ [47]. Therefore for $N \geq 2$, we consider the module $|1_k\rangle$ as a “vacuum”. The inner product (two-point function on $\mathcal{M}_{k,N}$) of $|1_k\rangle$ is given (defined) by the Nekrasov partition function (2.11):

$$\langle 1_k, 1_k \rangle = \langle 1_k | 1_k \rangle = \int_{\mathcal{M}_{k,N}} 1 = Z_{k,N}^{\text{Nek}}(\epsilon_1, \epsilon_2, \vec{a}), \quad (\text{D.6})$$

where $1 \in H_T^*(\mathcal{M}_{k,N})$. As a higher rank generalization of (C.4), it is natural to give the normalization of the inner product of $|Y\rangle$ in (D.2) as

$$\langle Y | Y' \rangle = \frac{(-1)^{|Y|-\ell_Y} c_Y}{(\epsilon_1 \epsilon_2)^{\ell_Y} |\text{Aut}(Y)| N \prod_I Y_I} \delta_{YY'} \times Z_{k-l,N}^{\text{Nek}}(\epsilon_1, \epsilon_2, \vec{a}), \quad (\text{D.7})$$

where c_Y is a constant which is not fixed in this paper. In [47], by Maulik and Okounkov it was shown that the operator of quantum multiplication by the divisor for $\mathcal{M}_{k,N}$ is given by the q -deformed Hamiltonian of a coupled N -tuple of Calogero-Sutherland system:¹⁵

$$\begin{aligned} H_D = & \frac{1}{2} \sum_{i=1}^N \sum_{m,n=1}^{\infty} (\epsilon_1 \epsilon_2 \alpha_{-m}^{(i)} \alpha_{-n}^{(i)} \alpha_{m+n}^{(i)} - \alpha_{-m-n}^{(i)} \alpha_m^{(i)} \alpha_n^{(i)}) \\ & + \sum_{i=1}^N \sum_{n=1}^{\infty} \left(a_i + \frac{\epsilon}{2} (1-n) \right) \alpha_{-n}^{(i)} \alpha_n^{(i)} - \epsilon \sum_{i < j} \sum_{n=1}^{\infty} n \alpha_{-n}^{(j)} \alpha_n^{(i)} \\ & - \epsilon \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} n \beta_{-n} \beta_n, \end{aligned} \quad (\text{D.8})$$

where $\epsilon = \epsilon_1 + \epsilon_2$. In [47], it was also shown that the divisor generates the quantum cohomology ring of $\mathcal{M}_{k,N}$. Note that in the $N = 1$ case, we need to modify the above operator H_D by adding

$$\frac{\epsilon q}{1-q} \sum_{n=1}^{\infty} \alpha_{-n} \alpha_n, \quad (\text{D.9})$$

¹⁵We have slightly changed the convention of [47].

and then by putting $a_1 = 0$, and $q \rightarrow -q$, the operator (C.5) for $\text{Hilb}_k(\mathbb{C}^2)$ is obtained.

In what follows, using the last term in (D.8):

$$H_D^{\text{quant}} \equiv -\epsilon \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} n \beta_{-n} \beta_n, \quad (\text{D.10})$$

and $\beta_n |1_k\rangle = 0$, we compute the quantum parts of two three-point functions related with the computations in section 4:

$$\langle D, Y_p, Y_p \rangle^{\text{qu}} \equiv \langle Y_p | H_D^{\text{quant}} | Y_p \rangle. \quad (\text{D.11})$$

Here $Y_1 \equiv \beta_{-1} 1_{k-1} \in H_T^{2N-2}(\mathcal{M}_{k,N})$, $Y_2 \equiv \beta_{-1}^2 1_{k-2} \in H_T^{4N-4}(\mathcal{M}_{k,N})$, and D is the Poincaré dual of the divisor class.¹⁶ By (D.7) and (D.10), we obtain

$$\langle D, Y_1, Y_1 \rangle^{\text{qu}} = -\frac{c_1 \epsilon}{\epsilon_1 \epsilon_2} L_0(q) Z_{k-1,N}^{\text{Nek}}(\epsilon_1, \epsilon_2, \vec{a}), \quad (\text{D.12})$$

$$\langle D, Y_2, Y_2 \rangle^{\text{qu}} = -\frac{c_2 \epsilon}{\epsilon_1^2 \epsilon_2^2} L_0(q) Z_{k-2,N}^{\text{Nek}}(\epsilon_1, \epsilon_2, \vec{a}), \quad (\text{D.13})$$

where $c_{1,2}$ are constants, and $L_0(q)$ is defined in (C.13). It would be interesting to determine the classical parts and fix the normalization constants.

E Exact Kähler potential for Calabi-Yau threefolds and fourfolds

In this appendix, we summarize the exact Kähler potentials on quantum Kähler moduli spaces of Calabi-Yau threefolds (e.g. [42]) and fourfolds (conjectured in [53]). The Kähler moduli space $\mathcal{M}_{\text{Kähler}}(X)$ of Calabi-Yau d -fold X is defined by $H^1(\wedge^1 T^* X)$, where $T^* X$ is the holomorphic cotangent bundle on X . By considering the NLSM propagating on X , the Kähler moduli space $\mathcal{M}_{\text{Kähler}}(X)$ is quantized by the α' corrections.

For Calabi-Yau threefold, it is known that around a large radius point the quantum-corrected Kähler potential K on $\mathcal{M}_{\text{Kähler}}(X)$ is given by

$$\begin{aligned} e^{-K} = & -\frac{i}{3!} \kappa_{ijk} (t^i - \bar{t}^i)(t^j - \bar{t}^j)(t^k - \bar{t}^k) + \frac{1}{4\pi^3} \zeta(3) \chi(X) \\ & - i \left[\frac{\partial}{\partial t^k} F(t) + \frac{\partial}{\partial \bar{t}^k} \overline{F(t)} \right] (t^k - \bar{t}^k) + 2i(F(t) - \overline{F(t)}), \end{aligned} \quad (\text{E.1})$$

where $i, j, k = 1, \dots, h^{1,1}(X)$. Here t^ℓ are the complexified Kähler parameters, κ_{lmn} are the classical triple intersection numbers of divisors on X , $\chi(X)$ is the Euler characteristic of X , and

$$F(t) = \frac{1}{(2\pi i)^3} \sum_{\beta \in H_2(X, \mathbb{Z}) \setminus \{0\}} n_\beta \text{Li}_3(q^\beta), \quad q^\beta \equiv e^{2\pi i \beta_k t^k} \quad (\text{E.2})$$

is the prepotential which gives the Gromov-Witten invariants n_β defined by the holomorphic maps $\partial\phi = 0$ in the (A-twisted) NLSM $\phi : \mathbb{P}^1 \rightarrow X$. Let \mathcal{O}_{J_i} be the observables

¹⁶For $N = 2$, we see that $Y_1 = D$.

associated with $J_i \in H^{1,1}(X)$, then the prepotential $F(t)$ can be obtained from the three-point function $\langle \mathcal{O}_{J_i} \mathcal{O}_{J_j} \mathcal{O}_{J_k} \rangle$ on \mathbb{P}^1 in the topological A-model via the relation

$$\langle \mathcal{O}_{J_i} \mathcal{O}_{J_j} \mathcal{O}_{J_k} \rangle = \kappa_{ijk} + \frac{\partial^3}{\partial t^i \partial t^j \partial t^k} F(t). \quad (\text{E.3})$$

For Calabi-Yau fourfold, it was conjectured that the Kähler potential K around a large radius point is given by [53]

$$\begin{aligned} e^{-K} = & \frac{1}{4!} \kappa_{ijkl} (t^i - \bar{t}^i)(t^j - \bar{t}^j)(t^k - \bar{t}^k)(t^\ell - \bar{t}^\ell) + \frac{i}{4\pi^3} \zeta(3) C_\ell (t^\ell - \bar{t}^\ell) \\ & + \frac{1}{2} (G_{k\ell}(t) + \overline{G_{k\ell}(t)})(t^k - \bar{t}^k)(t^\ell - \bar{t}^\ell) \\ & - (H_\ell(t) - \overline{H_\ell(t)})(t^\ell - \bar{t}^\ell) + \frac{1}{2} \eta^{mn} (F_{mn;\ell}(t) - \overline{F_{mn;\ell}(t)})(t^\ell - \bar{t}^\ell) \\ & - \frac{1}{2} \eta^{mn} (F_m(t) - \overline{F_m(t)})(F_n(t) - \overline{F_n(t)}), \end{aligned} \quad (\text{E.4})$$

where $i, j, k, \ell = 1, \dots, h^{1,1}(X)$ and $m, n = 1, \dots, h_{\text{prim}}^{2,2}(X)$.¹⁷ Here κ_{ijkl} are the classical quadruple intersection numbers of divisors on X , $C_\ell = \int_X c_3(X) \wedge J_\ell$ defined by the third Chern class $c_3(X)$ of X and $J_\ell \in H^{1,1}(X)$, η^{mn} is the inverse matrix of the intersection matrix $\eta_{mn} = \int_X H_m \wedge H_n$ on $H_{\text{prim}}^{2,2}(X)$. Similar to the case of Calabi-Yau threefold, the quantum corrections are given by the generating functions

$$F_n(t) = \frac{1}{(2\pi i)^2} \sum_{\beta \in H_2(X, \mathbb{Z}) \setminus \{0\}} n_{\beta, n} \text{Li}_2(q^\beta) \quad (\text{E.5})$$

which give the Gromov-Witten invariants $n_{\beta, n}$ defined by the holomorphic maps $\partial\phi = 0$ intersecting with the cycle dual to $H_n \in H_{\text{prim}}^{2,2}(X)$ in the (A-twisted) NLSM $\phi: \mathbb{P}^1 \rightarrow X$. The generating function $F_n(t)$ is obtained from the three-point function $\langle \mathcal{O}_{J_i} \mathcal{O}_{J_j} \mathcal{O}_{H_n} \rangle$ on \mathbb{P}^1 in the topological A-model via the relation

$$\langle \mathcal{O}_{J_i} \mathcal{O}_{J_j} \mathcal{O}_{H_n} \rangle = \kappa_{ijn} + \frac{\partial^2}{\partial t^i \partial t^j} F_n(t), \quad (\text{E.6})$$

where \mathcal{O}_{H_n} is the observable associated with $H_n \in H_{\text{prim}}^{2,2}(X)$, and $\kappa_{ijn} = \int_X J_i \wedge J_j \wedge H_n$ is the classical intersection number. In the conjectural formula (E.4), the generating functions

$$G_{k\ell}(t) = \frac{1}{(2\pi i)^2} \sum_{\beta \in H_2(X, \mathbb{Z}) \setminus \{0\}} n_{\beta, k\ell} \text{Li}_2(q^\beta) \quad (\text{E.7})$$

count the Gromov-Witten invariants $n_{\beta, k\ell}$ defined by the holomorphic maps $\partial\phi = 0$ intersecting with the cycle dual to $J_k \wedge J_\ell \in H_{\text{prim}}^{2,2}(X)$, and by definition these generating functions are written by a linear combination of $F_n(t)$. Other quantities in (E.4) are defined by

$$H_\ell(t) = \int_{i\infty}^{t^\ell} G_{\ell\ell}(t) dt^\ell + 2 \sum_{k \neq \ell} \int_{i\infty}^{t^k} G_{k\ell}(t) dt^k \Big|_{t^\ell = i\infty}, \quad (\text{E.8})$$

$$F_{mn;\ell}(t) = \int_{i\infty}^{t^\ell} \partial_\ell F_m(t) \partial_\ell F_n(t) dt^\ell. \quad (\text{E.9})$$

¹⁷ $h_{\text{prim}}^{2,2}(X)$ is the dimension of the primary subspace $H_{\text{prim}}^{2,2}(X) \subset H^{2,2}(X)$ whose elements are obtained from the wedge products of the elements of $H^{1,1}(X)$.

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References

- [1] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys. B* **426** (1994) 19 [Erratum *ibid.* **B 430** (1994) 485] [[hep-th/9407087](#)] [[INSPIRE](#)].
- [2] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys. B* **431** (1994) 484 [[hep-th/9408099](#)] [[INSPIRE](#)].
- [3] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N.P. Warner, *Selfdual strings and $N = 2$ supersymmetric field theory*, *Nucl. Phys. B* **477** (1996) 746 [[hep-th/9604034](#)] [[INSPIRE](#)].
- [4] S.H. Katz, A. Klemm and C. Vafa, *Geometric engineering of quantum field theories*, *Nucl. Phys. B* **497** (1997) 173 [[hep-th/9609239](#)] [[INSPIRE](#)].
- [5] S. Katz, P. Mayr and C. Vafa, *Mirror symmetry and exact solution of 4D $N = 2$ gauge theories: 1*, *Adv. Theor. Math. Phys.* **1** (1998) 53 [[hep-th/9706110](#)] [[INSPIRE](#)].
- [6] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2004) 831 [[hep-th/0206161](#)] [[INSPIRE](#)].
- [7] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Yu.I. Manin, *Construction of instantons*, *Phys. Lett. A* **65** (1978) 185 [[INSPIRE](#)].
- [8] G.W. Moore, N. Nekrasov and S. Shatashvili, *Integrating over Higgs branches*, *Commun. Math. Phys.* **209** (2000) 97 [[hep-th/9712241](#)] [[INSPIRE](#)].
- [9] G.W. Moore, N. Nekrasov and S. Shatashvili, *D particle bound states and generalized instantons*, *Commun. Math. Phys.* **209** (2000) 77 [[hep-th/9803265](#)] [[INSPIRE](#)].
- [10] H. Nakajima and K. Yoshioka, *Instanton counting on blowup. 1*, *Invent. Math.* **162** (2005) 313 [[math/0306198](#)] [[INSPIRE](#)].
- [11] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, [[hep-th/0306238](#)] [[INSPIRE](#)].
- [12] A. Braverman and P. Etingof, *Instanton counting via affine Lie algebras II: from Whittaker vectors to the Seiberg-Witten prepotential*, [[math/0409441](#)] [[INSPIRE](#)].
- [13] A. Iqbal and A.-K. Kashani-Poor, *Instanton counting and Chern-Simons theory*, *Adv. Theor. Math. Phys.* **7** (2004) 457 [[hep-th/0212279](#)] [[INSPIRE](#)].
- [14] A. Iqbal and A.-K. Kashani-Poor, *SU(N) geometries and topological string amplitudes*, *Adv. Theor. Math. Phys.* **10** (2006) 1 [[hep-th/0306032](#)] [[INSPIRE](#)].
- [15] T. Eguchi and H. Kanno, *Topological strings and Nekrasov's formulas*, *JHEP* **12** (2003) 006 [[hep-th/0310235](#)] [[INSPIRE](#)].
- [16] T.J. Hollowood, A. Iqbal and C. Vafa, *Matrix models, geometric engineering and elliptic genera*, *JHEP* **03** (2008) 069 [[hep-th/0310272](#)] [[INSPIRE](#)].
- [17] T. Eguchi and H. Kanno, *Geometric transitions, Chern-Simons gauge theory and Veneziano type amplitudes*, *Phys. Lett. B* **585** (2004) 163 [[hep-th/0312234](#)] [[INSPIRE](#)].
- [18] J. Zhou, *Curve counting and instanton counting*, [[math/0311237](#)] [[INSPIRE](#)].
- [19] A. Iqbal, *All genus topological string amplitudes and five-brane webs as Feynman diagrams*, [[hep-th/0207114](#)] [[INSPIRE](#)].

- [20] M. Aganagic, A. Klemm, M. Mariño and C. Vafa, *The topological vertex*, *Commun. Math. Phys.* **254** (2005) 425 [[hep-th/0305132](#)] [[INSPIRE](#)].
- [21] H. Awata and H. Kanno, *Instanton counting, Macdonald functions and the moduli space of D-branes*, *JHEP* **05** (2005) 039 [[hep-th/0502061](#)] [[INSPIRE](#)].
- [22] A. Iqbal, C. Kozcaz and C. Vafa, *The refined topological vertex*, *JHEP* **10** (2009) 069 [[hep-th/0701156](#)] [[INSPIRE](#)].
- [23] M. Taki, *Refined topological vertex and instanton counting*, *JHEP* **03** (2008) 048 [[arXiv:0710.1776](#)] [[INSPIRE](#)].
- [24] H. Awata and H. Kanno, *Refined BPS state counting from Nekrasov's formula and Macdonald functions*, *Int. J. Mod. Phys. A* **24** (2009) 2253 [[arXiv:0805.0191](#)] [[INSPIRE](#)].
- [25] F. Benini and S. Cremonesi, *Partition functions of $\mathcal{N} = (2, 2)$ gauge theories on S^2 and vortices*, *Commun. Math. Phys.* **334** (2015) 1483 [[arXiv:1206.2356](#)] [[INSPIRE](#)].
- [26] N. Doroud, J. Gomis, B. Le Floch and S. Lee, *Exact results in $D = 2$ supersymmetric gauge theories*, *JHEP* **05** (2013) 093 [[arXiv:1206.2606](#)] [[INSPIRE](#)].
- [27] G. Bonelli, A. Sciarappa, A. Tanzini and P. Vasko, *The stringy instanton partition function*, *JHEP* **01** (2014) 038 [[arXiv:1306.0432](#)] [[INSPIRE](#)].
- [28] R. Blumenhagen, M. Cvetič and T. Weigand, *Spacetime instanton corrections in 4D string vacua: the seesaw mechanism for D-brane models*, *Nucl. Phys. B* **771** (2007) 113 [[hep-th/0609191](#)] [[INSPIRE](#)].
- [29] L.E. Ibáñez and A.M. Uranga, *Neutrino Majorana masses from string theory instanton effects*, *JHEP* **03** (2007) 052 [[hep-th/0609213](#)] [[INSPIRE](#)].
- [30] B. Florea, S. Kachru, J. McGreevy and N. Saulina, *Stringy instantons and quiver gauge theories*, *JHEP* **05** (2007) 024 [[hep-th/0610003](#)] [[INSPIRE](#)].
- [31] R. Argurio, M. Bertolini, G. Ferretti, A. Lerda and C. Petersson, *Stringy instantons at orbifold singularities*, *JHEP* **06** (2007) 067 [[arXiv:0704.0262](#)] [[INSPIRE](#)].
- [32] M. Bianchi, F. Fucito and J.F. Morales, *D-brane instantons on the T^6/Z_3 orientifold*, *JHEP* **07** (2007) 038 [[arXiv:0704.0784](#)] [[INSPIRE](#)].
- [33] M. Billó, L. Ferro, M. Frau, L. Gallot, A. Lerda and I. Pesando, *Exotic instanton counting and heterotic/type-I-prime duality*, *JHEP* **07** (2009) 092 [[arXiv:0905.4586](#)] [[INSPIRE](#)].
- [34] F. Fucito, J.F. Morales and R. Poghossian, *Exotic prepotentials from $D(-1)D7$ dynamics*, *JHEP* **10** (2009) 041 [[arXiv:0906.3802](#)] [[INSPIRE](#)].
- [35] M. Billó, M. Frau, F. Fucito, A. Lerda, J.F. Morales and R. Poghossian, *Stringy instanton corrections to $N = 2$ gauge couplings*, *JHEP* **05** (2010) 107 [[arXiv:1002.4322](#)] [[INSPIRE](#)].
- [36] H. Ghorbani, D. Musso and A. Lerda, *Stringy instanton effects in $N = 2$ gauge theories*, *JHEP* **03** (2011) 052 [[arXiv:1012.1122](#)] [[INSPIRE](#)].
- [37] H. Ghorbani and D. Musso, *Stringy instantons in $SU(N)$ $N = 2$ non-conformal gauge theories*, *JHEP* **12** (2011) 070 [[arXiv:1111.0842](#)] [[INSPIRE](#)].
- [38] H. Ghorbani, *Gauge-stringy instantons in $\mathcal{N} = 2$ $U(N)$ gauge theories*, *JHEP* **12** (2013) 041 [[arXiv:1306.1487](#)] [[INSPIRE](#)].
- [39] Y. Tachikawa, *Five-dimensional Chern-Simons terms and Nekrasov's instanton counting*, *JHEP* **02** (2004) 050 [[hep-th/0401184](#)] [[INSPIRE](#)].
- [40] L. Gottsche, H. Nakajima and K. Yoshioka, *K-theoretic Donaldson invariants via instanton counting*, *Pure Appl. Math. Quart.* **5** (2009) 1029 [[math/0611945](#)] [[INSPIRE](#)].

- [41] A. Karch, D. Lüst and D.J. Smith, *Equivalence of geometric engineering and Hanany-Witten via fractional branes*, *Nucl. Phys. B* **533** (1998) 348 [[hep-th/9803232](#)] [[INSPIRE](#)].
- [42] H. Jockers, V. Kumar, J.M. Lapan, D.R. Morrison and M. Romo, *Two-sphere partition functions and Gromov-Witten invariants*, *Commun. Math. Phys.* **325** (2014) 1139 [[arXiv:1208.6244](#)] [[INSPIRE](#)].
- [43] J. Gomis and S. Lee, *Exact Kähler potential from gauge theory and mirror symmetry*, *JHEP* **04** (2013) 019 [[arXiv:1210.6022](#)] [[INSPIRE](#)].
- [44] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series volume 18, American Mathematical Society, U.S.A. (1999).
- [45] A. Givental, *Equivariant Gromov-Witten invariants*, [alg-geom/9603021](#).
- [46] T. Coates and A. Givental, *Quantum Riemann-Roch, Lefschetz and Serre*, [math/0110142](#).
- [47] D. Maulik and A. Okounkov, *Quantum groups and quantum cohomology*, [arXiv:1211.1287](#) [[INSPIRE](#)].
- [48] N. Caporaso, L. Griguolo, M. Mariño, S. Pasquetti and D. Seminara, *Phase transitions, double-scaling limit and topological strings*, *Phys. Rev. D* **75** (2007) 046004 [[hep-th/0606120](#)] [[INSPIRE](#)].
- [49] I. Grojnowski, *Instantons and affine algebras I: the Hilbert scheme and vertex operators*, [alg-geom/9506020](#) [[INSPIRE](#)].
- [50] H. Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, *Ann. Math.* **145** (1997) 379 [[alg-geom/9507012](#)].
- [51] A. Okounkov and R. Pandharipande, *Quantum cohomology of the Hilbert scheme of points in the plane*, [math/0411210](#).
- [52] V. Baranovsky, *Moduli of sheaves on surfaces and action of the oscillator algebra*, *J. Diff. Geom.* **55** (2000) 193 [[math/9811092](#)].
- [53] Y. Honma and M. Manabe, *Exact Kähler Potential for Calabi-Yau fourfolds*, *JHEP* **05** (2013) 102 [[arXiv:1302.3760](#)] [[INSPIRE](#)].
- [54] M. Bershadsky, C. Vafa and V. Sadov, *D-branes and topological field theories*, *Nucl. Phys. B* **463** (1996) 420 [[hep-th/9511222](#)] [[INSPIRE](#)].
- [55] P. Di Vecchia, A. Lerda and P. Merlatti, *$N = 1$ and $N = 2$ super Yang-Mills theories from wrapped branes*, *Nucl. Phys. B* **646** (2002) 43 [[hep-th/0205204](#)] [[INSPIRE](#)].
- [56] M.R. Douglas and G.W. Moore, *D-branes, quivers and ALE instantons*, [hep-th/9603167](#) [[INSPIRE](#)].
- [57] M.R. Douglas, *Enhanced gauge symmetry in M (atrix) theory*, *JHEP* **07** (1997) 004 [[hep-th/9612126](#)] [[INSPIRE](#)].
- [58] D.-E. Diaconescu, M.R. Douglas and J. Gomis, *Fractional branes and wrapped branes*, *JHEP* **02** (1998) 013 [[hep-th/9712230](#)] [[INSPIRE](#)].
- [59] D.S. Park and J. Song, *The Seiberg-Witten Kähler potential as a two-sphere partition function*, *JHEP* **01** (2013) 142 [[arXiv:1211.0019](#)] [[INSPIRE](#)].
- [60] R. Flume and R. Poghossian, *An Algorithm for the microscopic evaluation of the coefficients of the Seiberg-Witten prepotential*, *Int. J. Mod. Phys. A* **18** (2003) 2541 [[hep-th/0208176](#)] [[INSPIRE](#)].
- [61] U. Bruzzo, F. Fucito, J.F. Morales and A. Tanzini, *Multiinstanton calculus and equivariant cohomology*, *JHEP* **05** (2003) 054 [[hep-th/0211108](#)] [[INSPIRE](#)].

- [62] G. Bonelli, A. Sciarappa, A. Tanzini and P. Vasko, *Vortex partition functions, wall crossing and equivariant Gromov-Witten invariants*, *Commun. Math. Phys.* **333** (2015) 717 [[arXiv:1307.5997](#)] [[INSPIRE](#)].
- [63] I. Ciocan-Fontanine, M. Konvalinka and I. Pak, *Quantum cohomology of $\text{Hilb}_n(\mathbb{C}^2)$ and the weighted hood walk on the Young diagrams*, (2009).
- [64] H. Nakajima and K. Yoshioka, *Lectures on instanton counting*, [math/0311058](#).
- [65] K.A. Intriligator, D.R. Morrison and N. Seiberg, *Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces*, *Nucl. Phys. B* **497** (1997) 56 [[hep-th/9702198](#)] [[INSPIRE](#)].
- [66] H. Nakajima and K. Yoshioka, *Instanton counting on blowup. II. K-theoretic partition function*, [math/0505553](#) [[INSPIRE](#)].
- [67] H. Ooguri and C. Vafa, *Two-dimensional black hole and singularities of CY manifolds*, *Nucl. Phys. B* **463** (1996) 55 [[hep-th/9511164](#)] [[INSPIRE](#)].
- [68] M. Bershadsky, C. Vafa and V. Sadov, *D strings on D manifolds*, *Nucl. Phys. B* **463** (1996) 398 [[hep-th/9510225](#)] [[INSPIRE](#)].
- [69] O. Aharony, M. Berkooz and N. Seiberg, *Light cone description of $(2,0)$ superconformal theories in six-dimensions*, *Adv. Theor. Math. Phys.* **2** (1998) 119 [[hep-th/9712117](#)] [[INSPIRE](#)].
- [70] N. Nekrasov and A.S. Schwarz, *Instantons on noncommutative R^4 and $(2,0)$ superconformal six-dimensional theory*, *Commun. Math. Phys.* **198** (1998) 689 [[hep-th/9802068](#)] [[INSPIRE](#)].
- [71] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [[hep-th/9908142](#)] [[INSPIRE](#)].
- [72] D.L. Jafferis, *Topological quiver matrix models and quantum foam*, [arXiv:0705.2250](#) [[INSPIRE](#)].
- [73] M. Cirařici, A. Sinkovics and R.J. Szabo, *Cohomological gauge theory, quiver matrix models and Donaldson-Thomas theory*, *Nucl. Phys. B* **809** (2009) 452 [[arXiv:0803.4188](#)] [[INSPIRE](#)].
- [74] H. Awata and H. Kanno, *Quiver matrix model and topological partition function in six dimensions*, *JHEP* **07** (2009) 076 [[arXiv:0905.0184](#)] [[INSPIRE](#)].
- [75] A. Hurwitz, *Über die Anzahl der Riemann'schen Flächen mit gegebenen Verzweigungspunkten*, *Math. Ann.* **55** (1902) 53.
- [76] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz theory, and completed cycles*, *Ann. Math.* **163** (2006) 517 [[math/0204305](#)].
- [77] D.S. Kulyabov, Yu. P. Rybakov, G.N. Shikin and L.P. Yuschenko, *Kink - like configurations of interacting scalar, electromagnetic and gravitational fields*, [math-ph/9902011](#) [[INSPIRE](#)].
- [78] I. Goulden, D. Jackson and R. Vakil, *The Gromov-Witten potential of a point, Hurwitz numbers and Hodge integrals*, [math/9910004](#) [[INSPIRE](#)].
- [79] W.P. Li, Z. Qin and W. Wang, *Hilbert scheme intersection numbers, Hurwitz numbers, and Gromov-Witten invariants*, *Contemp. Math.* **392** (2005) 67.
- [80] A.S. Losev, A. Marshakov and N.A. Nekrasov, *Small instantons, little strings and free fermions*, [hep-th/0302191](#) [[INSPIRE](#)].